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AMERICAN MATHEMATICAL SOCIETY

EDITED BY

EARLE R. HEDRICK
DAVID R. CURTISS WILLIAM R. LONGLEY

WITH THE ASSISTANCE OF

HAROLD T. DAVIS HARRY W. KUHN
THEOPHIL H. HILDEBRANDT DERRICK N. LEHMER
JOHN R. KLINE HENRY L. RIETZ
CAROLINE E. SEELY

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OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

THIRTY-NINTH SUMMER MEETING, CHICAGO, June 19-24, 1933.

This meeting will be in conjunction with the meeting of the American Association for the Advancement of Science at the Century of Progress Exposition, but most of the actual sessions will be at the University of Chicago. The Mathematical Association of America will participate, as usual, and will hold separate meetings on Tuesday, Thursday, and Saturday. A dinner of mathematicians and their guests is scheduled for Friday evening. Headquarters for mathematicians will be at Judson Court, a dormitory of the University of Chicago, at 60th St. and Ellis Avenue.

Two symposia have been arranged. At one of these, devoted to geometry, Professors Tullio Levi-Civita and Enrico Bompiani of the University of Rome, Professor W. C. Graustein of Harvard University, and others, will speak. At the other symposium, which will be devoted to analysis, Professor Lipót Féjer of the University of Budapest, Professor Dunham Jackson of the University of Minnesota, Professor C. N. Moore of the University of Cincinnati, and others, will speak. Professor Féjer's topic will be *The infinite sequences arising in the theories of harmonic analysis, of interpolation, and of mechanical quadratures.* There will be several additional invited addresses during the week. Professor Levi-Civita will present a paper entitled *On some mathematical aspects of the new mechanics*, and, at the same session, Professor G. D. Birkhoff of Harvard University will speak on *Quantum mechanics and asymptotic series.* Professor L. E. Dickson of the University of Chicago will give an address entitled *Recent progress in additive number theory.*

Professors Levi-Civita, Bompiani, and Féjer are the guests of the Century of Progress Exposition and the American Association for the Advancement of Science. The meetings at which they speak, and certain others, will be joint sessions of the American Mathematical Society and Section A of the A.A.A.S.

Titles and abstracts of papers to be presented at this meeting should be in the hands of Associate Secretary Mark H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than May 20. Abstracts received before May 6 will appear in the May issue of this BULLETIN.

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to the American Mathematical Society, 450-459 Ahnaip St., Menasha, Wis., or 501 West 116th St., New York.

The initiation fees and the annual dues of members of the Society (see this BULLETIN, p. 322, May, 1930; and the List of Officers and Members, September, 1932, p. 66), are payable to the Treasurer of the Society, Professor G. W. Mullins, 501 West 116th St., New York City.

THE FEBRUARY MEETING IN NEW YORK

The three hundred third meeting of the American Mathematical Society was held at Columbia University, on Saturday, February 25, 1933, extending through the usual morning and afternoon sessions. The attendance included the following ninety members of the Society:

R. L. Anderson, R. G. Archibald, H. E. Arnold, J. L. Barnes, R. A. Beaver, M. F. Becker, Samuel Borofsky, A. B. Brown, E. W. Brown, A. D. Campbell, Alonzo Church, George Comenetz, R. P. Conkling, M. F. Deuring, J. L. Doob, Jesse Douglas, Arnold Dresden, Lillian Dundes, H. T. Engstrom, J. M. Feld, T. S. Fiske, W. B. Fite, D. A. Flanders, Edward Fleisher, M. M. Flood, R. M. Foster, T. C. Fry, C. A. Garabedian, R. M. Gut, E. H. Hanson, Alan Hazeltine, Lulu Hofmann, Harold Hotelling, J. C. Hughes, Nathan Jacobson, S. A. Joffe, I. N. Kagno, Edward Kasner, J. R. Kline, Morris Kline, M. S. Knebelman, B. O. Koopman, A. W. Landers, M. K. Landers, W. H. Langdon, Harry Langman, Solomon Lefschetz, A. K. Leon, H. C. Levinson, A. J. Lotka, L. A. MacColl, H. F. MacNeish, A. E. Meder, A. K. Mitchell, E. C. Molina, G. W. Mullins, C. A. Nelson, E. P. Northrop, Oystein Ore, F. W. Owens, R. E. A. C. Paley, Walter Prenowitz, G. B. Price, W. C. Randels, H. W. Raudenbush, L. M. Reagan, M. S. Rees, H. L. Rietz, J. F. Ritt, S. L. Robinson, C. E. Seely, F. R. Sharpe, Jacob Sherman, L. P. Siceloff, L. L. Smail, P. A. Smith, M. H. Stone, Paul Szabo, H. B. Thiessen, J. M. Thomas, E. W. Titt, I. L. Van Dyck, Oswald Veblen, H. E. Wahlert, R. J. Walker, Henry Walther, M. E. Wells, R. A. Wetzel, W. A. Wilson, Leo Zippin.

At a business session of the Society held before the beginning of the afternoon session the following changes in the By-Laws were adopted, as recommended by the Council: Article I, Section 2, to read "It shall be a duty of the President to deliver an address before the Society at the close of his term of office or within one year thereafter." Article VI, Section 2, last clause of the first sentence, to read "four dollars of the dues of each member shall be for a year's subscription to the Bulletin."

At the beginning of the afternoon session, by invitation of the Program Committee, Professor F. R. Sharpe, of Cornell University, delivered an address entitled *The algebraic theory of involutorial transformations.*

Titles and cross-references to the abstracts of the papers read at the regular sessions follow below. Professor W. A. Wilson presided at the morning session, and Professor Arnold Dresden in the afternoon. Mr. Halperin was introduced by Professor

Kasner, and Mr. Richardson by Professor P. A. Smith. The papers whose abstract numbers are followed by the letter *t* were read by title.

1. *Pseudo-covariants of an n-ic in m variables in a Galois field that consist of terms of this n-ic*, by Professor A. D. Campbell. (Abstract No. 39–3–67.)

2. *The non-analytic character and the areas of Kasner's convex curves*, by Mr. Morris Halperin. (Abstract No. 39–3–68.)

3. *The ranges of analytic functions*, by Dr. J. L. Doob (National Research Fellow). (Abstract No. 39–3–69.)

4. *Finite plane euclidean geometry*, by Mr. R. A. Beaver. (Abstract No. 39–3–70.)

5. *Cell-spaces*, by Dr. A. W. Tucker (National Research Fellow). (Abstract No. 39–3–71–*t*.)

6. *Invariants of symmetric product complexes*, by Mr. Moses Richardson. (Abstract No. 39–3–72.)

7. *The torsion coefficients of symmetric products*, by Professor P. A. Smith. (Abstract No. 39–3–73.)

8. *A separation theorem*, by Professor W. A. Wilson. (Abstract No. 39–3–74.)

9. *A metrization theorem of Chittenden and Pitcher*, by Dr. Selby Robinson (National Research Fellow). (Abstract No. 39–3–75.)

10. *On a special class of polynomials*, by Professor Oystein Ore. (Abstract No. 39–3–76.)

11. *A note on compactness*, by Mr. E. H. Hanson. (Abstract No. 39–3–77–*t*.)

12. *Concurrence and uncountability*, by Dr. N. E. Rutt. (Abstract No. 39–3–78–*t*.)

13. *Prime ends and order*, by Dr. N. E. Rutt. (Abstract No. 39–3–79–*t*.)

14. *Sets of local separating points of a continuum*, by Dr. G. T. Whyburn. (Abstract No. 39–3–80–*t*.)

15. *On primary normal division algebras of degree eight*, by Professor A. A. Albert. (Abstract No. 39–1–65–*t*.)

16. *Cyclic fields of degree eight*, by Professor A. A. Albert. (Abstract No. 39–1–66–*t*.)

17. *A note on the equivalence of algebras of degree two*, by Professor A. A. Albert. (Abstract No. 39–3–81–*t*.)

18. *On Young's definition of an algebra*, by Dr. L. E. Bush. (Abstract No. 39–3–82–*t*.)

19. *On a generalization of certain theorems on algebras with a modulus*, by Dr. L. E. Bush. (Abstract No. 39–3–83–*t*.)

20. *The representation of integers as sums of pyramidal numbers*, by Dr. R. D. James (National Research Fellow). (Abstract No. 39–3–84–*t*.)

21. *An application of characteristic functions to statistics* (preliminary report), by Mr. Solomon Kullback. (Abstract No. 39–3–85–*t*.)

22. *On the structure of Boolean algebras*, by Professor M. H. Stone. (Abstract No. 39–3–86–*t*.)

23. *Bloch's theorem for minimal surfaces*, by Dr. E. F. Beckenbach (National Research Fellow). (Abstract No. 39–3–87–*t*.)

24. *A third-order irregular boundary value problem and the associated series*, by Professor L. E. Ward. (Abstract No. 39–3–88–*t*.)

25. *Notes on the theory and application of Fourier transforms*, I–II, by Dr. R. E. A. C. Paley and Professor Norbert Wiener. (Abstract No. 39–3–89–*t*.)

26. *On the decomposition of prime ideals in relative icosahedron fields*, by Dr. R. M. Gut (Rockefeller Foundation Fellow). (Abstract No. 39–3–90.)

27. *A Jordan space-curve which bounds no finite simply-connected area*, by Professor Jesse Douglas. (Abstract No. 39–3–91.)

28. *On the division transformation for matric polynomials*, by Mr. M. M. Flood. (Abstract No. 39–3–92.)

29. *The uniformly loaded thick rectangular plate with at least two opposite edges supported*, by Professor C. A. Garabedian. (Abstract No. 39–3–93.)

30. *A study of certain dynamical systems with applications to the generalized double pendulum*, by Dr. G. B. Price. (Abstract No. 39–3–94.)

J. R. KLINE, *Associate Secretary*

THE MARCH MEETING IN PALO ALTO

The three hundred fourth meeting of the American Mathematical Society was held at Stanford University, on Saturday, March 18, 1933. There was a morning and an afternoon session. The attendance included the following twenty-six members:

A. D. B. Andrews, H. M. Bacon, E. T. Bell. B. A. Bernstein, H. F. Blichfeldt, Thomas Buck, F. A. Butter, C. E. Corbin, D. C. Duncan, O. G. Harrold, M. A. Heaslet, R. D. James, Vern James, D. N. Lehmer, C. F. Luther, W. A. Manning, R. G. Mason, A. D. Michal, F. R. Morris, S. L. Parker, T. S. Peterson, A. E. Ross, Pauline Sperry, J. V. Uspensky, A. R. Williams, B. C. Wong.

There was no meeting of the Council. By invitation of the Program Committee, Professor Pauline Sperry delivered an address entitled *Ernest Julius Wilczynski*. Professor D. N. Lehmer presided at both sessions.

Titles and cross-references to the abstracts of the papers read at this meeting follow below. The papers whose abstract numbers are followed by the letter *t* were read by title. Dr. Diamond was introduced by Professor Bernstein.

1. *Concerning primitive groups of class u.* Paper II, by Dr. C. F. Luther. (Abstract No. 39–3–95.)

2. *Ernest Julius Wilczynski*, by Professor Pauline Sperry. (Abstract No. 39–3–96.)

3. *Simplification of the set of four postulates for Boolean algebras in terms of rejection*, by Professor B. A. Bernstein. (Abstract No. 39–3–97.)

4. *The complete existential theory of the Whitehead-Huntington set of postulates for the algebra of logic*, by Dr. A. H. Diamond. (Abstract No. 39–3–98.)

5. *The Latin square functions*, by Professor E. T. Bell. (Abstract No. 39–3–99.)

6. *A note on three equivalent theorems*, by Dr. A. E. Ross (National Research Fellow). (Abstract No. 39–3–100.)

7. *A self-dual septimic possessing seven of each kind of the simple singularities, autopolar by seven rectangular hyperbolas and a circle*, by Dr. D. C. Duncan. (Abstract No. 39–3–101.)

8. *Three autopolar rational quintic curves*, by Dr. D. C. Duncan. (Abstract No. 39–3–102.)

9. *The value of the number $g(k)$ in Waring's problem*, by Dr.

R. D. James (National Research Fellow). (Abstract No. 39–3–103.)

10. *The space quartic of the second kind by synthetic methods*, by Dr. Anne D. B. Andrews. (Abstract No. 39–3–104.)

11. *An existence theorem for differential systems*, by Professor W. M. Whyburn. (Abstract No. 39–3–105–*t*.)

12. *On the representation of numbers modulo m*, by Mr. E. D. Rainville. (Abstract No. 39–3–106–*t*.)

13. *An integral equation with symmetric kernels*, by Dr. T. S. Peterson. (Abstract No. 39–3–107.)

14. *The analogue of Weyl's conformal curvature tensor in a Michal functional geometry*, by Dr. T. S. Peterson. (Abstract No. 39–3–108.)

15. *The order of primitive groups* (V), by Professor W. A. Manning. (Abstract No. 39–3–109.)

16. *The transformation* $y=f'(x)$, by Professor Raymond Garver. (Abstract No. 39–3–110–*t*.)

17. *On Section A of Principia Mathematica*, by Professor B. A. Bernstein. (Abstract No. 39–3–111–*t*.)

18. *Analogues of the Steiner surface and their double curves*, by Professor A. R. Williams. (Abstract No. 39–3–112.)

B. A. BERNSTEIN, *Acting Associate Secretary*

ORMOND STONE—1847–1933

Professor Ormond Stone, founder and first editor of the Annals of Mathematics, was instantly killed by an automobile on January 17, 1933, while he was walking along the road near his home not far from Fairfax, Virginia.

Professor Stone, who was eighty-six years old, was retired on the Carnegie Foundation in 1912, after serving twenty years as Professor of Astronomy and Director of the Leander McCormick Observatory at the University of Virginia. Before coming to the University of Virginia he had been an assistant at the U. S. Naval Observatory and afterwards Director of the Cincinnati Observatory. He observed the total eclipse of the sun in Iowa in 1869; was in charge of the Naval Observatory eclipse expedition to Colorado in 1878, and of the McCormick Observatory expedition to South Carolina in 1900. While he was at the Cincinnati Observatory he played an important part in bringing about the adoption of the standard time belts.

A number of Professor Stone's former students and assistants at the Leander McCormick Observatory now hold prominent executive and scientific positions in the United States. Among them are: Edgar O. Lovett, President of Rice Institute, Houston, Texas; Heber D. Curtis, Director of the University of Michigan Observatory; Charles P. Olivier, Director of the Flower Observatory, University of Pennsylvania; Herbert R. Morgan, U. S. Naval Observatory; Ralph E. Wilson, Dudley Observatory, Albany, N. Y.; G. F. Paddock, Lick Observatory; and T. McN. Simpson, Randolph-Macon College, Ashland, Va.

While at the University of Virginia Professor Stone was actively interested in the development of secondary education in the state, and was a member of the first Board of Visitors of the State Normal School at Harrisonburg.

Undoubtedly his major contribution to mathematics was the founding of the Annals of Mathematics. Soon after coming to the University of Virginia Professor Stone fulfilled a long-cherished desire to establish a journal of mathematics. He felt that the American Journal of Mathematics, newly founded at Johns Hopkins University by Sylvester, while admirably supplying a needed opportunity for publishing advanced work, did not furnish a vehicle for the publication of papers of intermediate difficulty and more popular character, and that there was a real need for such a publication. In 1884 he founded the Annals of Mathematics, supporting its publication financially from his private income for about twelve years. He succeeded in sustaining this journal in a creditable and dignified manner. He was assisted in the editorial work by a number of mathematical friends, among whom latterly was Professor Maxime Bôcher, of Harvard University. He was profoundly and unselfishly interested in his journal and it was with much distress that he was finally forced to withdraw his financial support.

The University of Virginia then undertook to support it financially for two years; but, that aid being withdrawn, the Annals was given to Harvard. After meeting there the same fate as at Virginia, it finally found a permanent home at Princeton. It was at this time of his relinquishing control of the Annals that the New York Mathematical Society, and its Bulletin, were established, to be

quickly succeeded by the American Mathematical Society, whose publications made a privately supported journal less necessary. This consoled Professor Stone in giving up the publication of the Annals.

American students in mathematics owe Professor Stone a real debt of gratitude for his earnest effort in helping to create an interest in their subject.

J. J. LUCK

HILBERT'S INTUITIONAL GEOMETRY

Anschauliche Geometrie. By D. Hilbert and S. Cohn-Vossen. Berlin, Julius Springer, 1932. vii+310 pp.+330 figures. R.M. 25.80.

The foundation of this extremely interesting book is a course of lectures delivered by Hilbert at the University of Göttingen in the winter of 1920–21. The notes were taken by W. Rosemann and supplemented and edited by S. Cohn-Vossen.

In the preface Hilbert points out that in mathematics as in all other scientific research we meet two sorts of tendencies: one towards abstraction, the other towards intuition. The first seeks to work out the logical points of view from the extensive material and to connect it systematically; the second strives for an intuitional conception and an understanding of relations of content.

The first has led to the magnificent systematic theories embodied in algebraic geometry, Riemannian geometry, and topology. Nevertheless, Hilbert maintains the position that intuitional geometry is still of great importance as a superior force of research and for the appreciation of the results of research.

The reviewer may be permitted to illuminate this by a personal experience. It is well known that the sextic of genus four, as the intersection of two generically located cubic and quadric surfaces, admits of 120 tritangent-planes. Just as in the case of the 28 double tangents of a general plane quartic, which may all be real, the question arises whether all 120 tritangent-planes may be real. I know of no place where this question has been answered nor of any method by which this problem could be solved. But I constructed a model of this sextic which shows the possibility of 120 real tritangent-planes. This, of course, is no mathematical proof.

Thus, in this, as in many other instances of mathematical research, it is obvious that intuitional aid by the way of construction of graphs and models and intuitional interpretation is in many cases very desirable and helpful.

Throughout the book the reader is struck by the loftiness of the standpoint from which the problems are viewed. One would of course expect this from a superior mathematical mind like Hilbert's.

The first chapter deals with the simplest curves and surfaces, conics and quadrics with their most striking characteristic properties.

In the second chapter are considered regular point systems in the plane and in space. The study of lattice-works has in recent years become of great importance not only for the proper comprehension of crystallographic systems but also for certain branches of number theory as appears from the memorable investigations of Minkowski. This is shown by such examples as the Leibniz

series $\pi/4 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$, the groups of plane and spatial movements and their application to the classification of crystals, regular polyhedrons, etc.

One of the most fascinating chapters of mathematics, in particular of geometry, is that of configurations and their connection with various geometries and finite group theory. Here, in our book, we find a discussion of the (7,3), (8,3), (9,3)-configurations, the theorems of Desargues and Pascal and their position in projective geometry, the configuration of the centers of four spheres and their centers of similitude, regular solids and cells in four-dimensional space and their projections, an account of enumerative methods in geometry, and Schläfli's double-sixes on the cubic surface. One may expect that differential geometry will play an important part in a book on intuitional geometry. This we find in Chapter IV, under such topics as differential properties of plane and space curves, curvature, developable and ruled surfaces, curves and surfaces of constant breadth, the principal properties of the sphere ("about which a whole book might be written"), cyclides, non-euclidean geometries and some of their models, circular transformations and stereographic projection, the partition of the complex plane by circular arcs and connected groups.

Next follows a chapter on kinematics, dealing with movements in the plane and in space, with Peaucellier's cell and other linkages, and well known curves generated by moving points subjected to certain mechanical conditions.

In Chapter VI we find a very interesting introduction to topology, a branch of mathematics of relatively recent origin, whose importance in the evolution of modern mathematics has become more and more apparent.

The discussion first centers about Euler's theorem and its generalization and the connectivity of surfaces. Then follows a rather detailed study of unilateral surfaces as exemplified by the projective plane and its various models. As such are presented various models constructed at Göttingen, Steiner's famous quartic surface (Römerfläche) and Boy's surface, for which no algebraic equivalent has been found so far.

Among the particularly fascinating problems of topology are those concerned with neighboring regions, as the four-color problem, and the dualistic "thread-problem" (Fadenproblem). As the latter is not as well known it is perhaps of interest to many readers to have it stated: "Given a surface; required the maximum number of points on it which may all be connected to each other by curves on the surface such that no two curves intersect each other." In the popular sense curve is identified with thread. As is well known, the four-color problem, the possibility of coloring any geographic map with four colors so that no two adjacent regions have the same color, has not yet been shown to have a solution in every possible case. It is one of those famous unsolved problems (like some in number theory) whose simplicity is such that any person with ordinary intelligence can grasp its content.

The book is clearly written and beautifully illustrated, and from a didactic standpoint will do a great deal of good. It will be thoroughly enjoyed by analyst and geometrician alike.

ARNOLD EMCH

OSGOOD ON FUNCTION THEORY, II, 2

Lehrbuch der Funktionentheorie. By W. F. Osgood. Zweiter Band, zweite Lieferung. Leipzig, Teubner, 1932. 9+377 pp.

This second section of Volume 2 of Osgood's well known treatise on function theory is devoted to Abelian integrals and periodic functions. The pagination and chapter numbering are continued from the first "Lieferung." The material here treated falls into five chapters, namely: IV. Algebraic functions and algebraic integrals. V. The existence theorem. The integrals and the prime functions. VI. Abel's theorem. Intersection theorems. Theorem of Riemann-Roch. VII. Periodic functions. VIII. Applications. These titles suggest their own story. There is, however, no steady progress from one chapter to the next since one may think of the behavior of the normal integrals upon the dissected Riemann surface as being the essential topic in each of the chapters. (Tables of periods are displayed no less than 45 times.) In Chapter IV (the first of this "Lieferung") the algebraic form $G(w, z)=0$ is treated both as a plane curve and as a Riemann surface. The discussion is confined as is fitting to cases in which multiple points are of the simplest type. The integrals are set up, their properties studied, the periods are normalized, the bilinear period relations of Weierstrass and of Riemann are established and compared. The treatment in this chapter is the traditional one save for more than usual (but much needed) discussion of the number of intersections of algebraic curves in the projective plane and an exposition of the theory of the plane of function theory. In Chapter V, the methods of logarithmic potentials are applied. Prime functions are introduced and the normal integrals are thereby expressed by use of differentiation. The conformal mapping established by inversion provides for an automorphic group whose invariants interpret the functions upon the Riemann surface. One returns to geometry in the study of Noether's normal curve and of its projections. The abstract treatment in this chapter and the extensive list of necessary special notations would make the reading at best difficult to most students. The avoidance of illustrative material tends to render the topic unnecessarily forbidding even although the reader will appreciate the higher view points provided for topics mentioned in the previous chapter. Chapter VI is devoted to Abel's theorem and the theorem of Riemann-Roch, both suggested earlier. The essentially new topics concern the use of homogeneous differential expressions. The adjoints of order $m-3$ are again discussed. The prime function is studied in the homogeneous form. There is additional material on space curves. As with other chapters illustrative material is scanty unless one is to view the entire chapter as essentially a review of the two preceding in terms of homogeneous variables. Chapter VII reverses the historical sequence and returns to the point of view of Jacobi. It approaches the problem of inversion, by the study of periodic functions in p arguments. A third of the chapter is covered by the normalization of the periods and related topics. Then the theta-functions are introduced as infinite series. A brief mention is made of characteristics, and the problem of normalizing the periods of the theta-functions is treated. Chapter VIII takes up the correspondence

principle for points on an algebraic curve, and the 28 double tangents of a quartic, and closes with five pages on the Riemann-Weierstrass theta-theorem.

There is little new in this book beyond the occasional use of the plane of function theory, nor is it encyclopedic. Although much of the language suggests geometry, mention is made only of those geometrical concepts which are strictly essential to the topic. The domain of variation is sometimes viewed as a curvilinear polygon, sometimes as a dissected Riemann surface, sometimes as a plane curve, sometimes as a curve in n-space. It is never the mere abstract bearer of algebraic groups of points, as handled successfully by the modern Italians. The pervading points of view are those of Klein with wordiness excised and with more explicit proofs. Charm and thrill are sometimes found in proceeding from the simple to the advanced, from the concrete to the abstract, in the developing conviction of power to handle problems, in catching glimpses of domains yet awaiting the adventurer. These qualities often apparent in the original papers of the investigators, and retained in such expositions as that of Appell and Goursat, are here consciously sacrificed to an impersonal economy in abstract and logical treatment which may be justified if this interesting subject is to be compressed to fit into a general treatise on function theory.

A. A. BENNETT

VEBLEN-WHITEHEAD—FOUNDATIONS OF DIFFERENTIAL GEOMETRY

The Foundations of Differential Geometry. By Oswald Veblen and J. H. C. Whitehead. (Cambridge Tracts in Mathematics and Mathematical Physics, No. 29.) Cambridge, at The University Press, 1932. New York, The Macmillan Company. ix+96 pp.

In the preface to his earlier work on *Invariants of Quadratic Differential Forms* (Cambridge, 1927), Professor Veblen regretfully stated that even a short discussion of differential geometry had been crowded out because of the limited space available. If the necessity for this omission led to the writing of the present tract, it was a fortunate circumstance, for the result is a profoundly stimulating book.

The title, *The Foundations of Differential Geometry*, is scarcely broad enough to describe the contents, for the book is a critical study of the foundations of all mathematical systems which have been called geometries. The authors have kept strictly to their theme, namely the logical foundations of geometry, and have not yielded to the temptation to develop any particular geometry much beyond the postulates by which it is defined. While the treatment is philosophical and highly critical, yet it is so skillfully done that the book is easy to read—a fact which testifies to the genius of the authors for mathematics of this kind.

A rigid definition of geometry is not attempted, on the ground that any objective definition of geometry should include the whole of mathematics. The question is dismissed with the statement that a branch of mathematics is called a geometry because the name seems good, on emotional and traditional

grounds, to a sufficient number of competent people. It is well known that in recent years geometry has outgrown the limitations placed upon it by Klein in his Erlanger Programm of 1870. Without attempting to be dogmatic, the authors give systems of axioms which they believe provide an adequate foundation for any of the differential geometries which are now being studied.

First the arithmetic space of n dimensions is discussed. An arithmetic point is merely an ordered set of n real numbers. The set of points linearly dependent on $k+1$ linearly independent points constitutes an arithmetic linear k-space. The usual theorems on systems of linear homogeneous equations are developed in geometric terminology.

The translation group, the centered affine group, the affine group, the orthogonal group, and the euclidean metric group are defined when operating upon the points of an arithmetic n-space, and the corresponding geometries are defined according to the idea of Klein.

Another specification of a geometry, namely as a *coordinate geometry*, is now discussed. A coordinate system is defined as a correspondence $P \rightarrow x$ between a set $[P]$ of points (undefined elements of the geometry) and a set of arithmetic points $[x]$. Another set of undefined elements are termed *preferred coordinate systems*. Four axioms relating these undefined elements and the group G of transformations of arithmetic points into arithmetic points establish the geometry of the group G.

The axiomatic approach to a differential geometry is less simple. An important concept introduced in this connection is that of *pseudo-group*. A set of transformations constitute a pseudo-group if (i) the resultant of two transformations is in the set provided it exists, and if (ii) the set contains the inverse of each transformation of the set. The word *geometric object* is preferred by the authors to the term *invariant* when invariance under a pseudo-group is intended. A geometric object in a simple manifold determines a structure and therefore a space. The geometry of this space is the geometry of the object. Geometric objects (and the spaces which they determine) are classified by means of the pseudo-group of regular point transformations. This classification of geometric objects is a natural generalization of Klein's point of view, a geometry defined by a geometric object falling within the categories of the Erlanger Programm if and only if the geometric object is characterized by its group of automorphisms.

A function of n variables, defined for all points of a region X, is of class u if it and its derivatives of orders $\leqq u$ exist and are continuous at each point of X. A transformation $y^i = y^i(x)$ is of class u if each of the n functions $y^i(x)$ is of class u. An allowable coordinate system of class u in a simple manifold of class $u' \geqq u$ is defined by the conditions:

1. If $[P]$ is the image in a preferred coordinate system K of an arithmetic region $[x]$, the correspondence $P \rightarrow x$ determined by K is an allowable coordinate system.

2. If $[x]$ is an arithmetic region, $P \rightarrow x$ an allowable coordinate system, and $x \rightarrow y$ a regular transformation of class u, then the resultant transformation $P \rightarrow y$ is an allowable coordinate system.

Chapter IV is concerned with the theory of k-cells in n-space and the local structure or infinitesimal geometry at a point P. Chapter V is on tangent

spaces in general, tangent and osculating Riemannian spaces, affine connections, and parallel and affine displacements. This chapter contains a very interesting critical examination of the foundations of tensor analysis.

Chapter VI, entitled *A Set of Axioms for Differential Geometry*, is the heart of the book. The axioms are stated in terms of *points* and *allowable coordinate systems*, points being undefined, and the allowable coordinate systems constituting an undefined class of biunique correspondences $P \rightarrow x$ between sets of points and sets of arithmetic points in the arithmetic n-space.

The first group of axioms describe the local structure completely:

A_1. The transformation of coordinates between two allowable coordinate systems which have the same domain is regular, provided one of them, at least, has a region for its arithmetic domain.

A_2. Any coordinate system obtained by a regular transformation of coordinates from an allowable coordinate system is allowable.

A_3. The correspondence in which each point of an n-cell corresponds to its image in an allowable coordinate system is an allowable coordinate system.

Let $[P]$ and $[Q]$ be the domains of biunique coordinate systems $P \rightarrow x$ and $Q \rightarrow y$ respectively, $[x]$ and $[y]$ their arithmetic domains, and $[R]$ the intersection of $[P]$ and $[Q]$. If the correspondences are consistent for all points of $[R]$, they define together a coordinate system called their *union*.

The second group of axioms characterizes the class of allowable coordinate systems in a space satisfying the axioms A:

B_1. Any coordinate system which is the union of a set of allowable coordinate systems whose domains are n-cells, is allowable.

B_2. Each allowable coordinate system is the union of a set of allowable coordinate systems whose domains are n-cells.

The third group of axioms are of a topological nature:

C_1. If two n-cells have a point in common, they have in common an n-cell containing this point.

C_2. If P and Q are any two distinct points, there is an n-cell C_P containing P, and an n-cell C_Q containing Q, such that C_Q has no point in common with C_P.

C_3. There exist at least two points.

A surprising number (seventeen) of non-trivial theorems are deduced from these seemingly mild axioms, as, for instance, the result that the totality of regular point transformations between regions in a regular manifold is a pseudo-group.

However, it is noted that the axioms describe a large class of spaces which are not all equivalent, so that further special postulates must be added to secure special geometries. Some of these are considered in the last chapter.

As Mr. Mencken would conclude, "There is an index."

C. C. MacDUFFEE

VOLUME I—ERGEBNISSE DER MATHEMATIK

Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 1, 1932.

Knotentheorie. By K. Reidemeister, 74 pp., 114 figures.

Graphische Kinematik und Kinetostatik. By K. Federhofer. 112 pp., 27 figures.

Lamésche-, Mathieusche- und verwandte Funktionen in Physik und Technik. By M. J. O. Strutt. 116 pp., 12 figures.

Die Methoden zur angenäherten Losung von Eigenwertproblemen in der Elastokinetik. By K. Hohenemser. 89 pp., 15 figures.

Fastperiodische Funktionen. By H. Bohr. 96 pp., 10 figures.

This series of reports forms a welcome addition to the work of abstracting published by the Zentralblatt für Mathematik and is being edited by the directors of this journal. The need of such reports is becoming more and more evident as scientific knowledge expands and it is particularly pleasing that this volume contains an astonishing amount of information in a compact form and presented in good style. The reports deal mainly with recent work and so supplement the articles in the Encyklopädie der Mathematischen Wissenschaften in which the history of each subject is given.

A reader who compares Reidemeister's modern account of the theory of knots with the early accounts given by Listing and Tait will realize that the subject has been greatly transformed by the use of groups, matrices, quadratic forms, and polynomials. A knot is regarded here primarily as a skew polygon. This plan simplifies the description of deformations and operations; it also facilitates the drawing of a regular projection of the knot, on a plane P. Such a projection divides P into regions which may be divided into black and white regions by a suitable method of coloring. Special names are used when there are only two or only three black regions.

The study of topological invariants, such as torsion numbers, forms an important part of the subject and it is interesting to note the substantial contributions which have been made by the American writers J. W. Alexander and G. B. Briggs. Reidemeister's report is well written and unlike many recent articles on topology does not make the subject look formidable.

Federhofer's report shows that pure geometry is still quite a live subject capable of interesting and useful applications in the study of driving mechanisms and linkages. The author has been particularly active in inventing new constructions for the elucidation of the motion of a plane system in a plane and the motion of a rigid system in space. He gives a fine account of the theory of kinematic chains and linkages, calling attention to the forgotten work of Grübler. The whole report is, indeed, furnished with very complete lists of references.

In kinetostatics the chief aim is to determine the guiding forces, linkage pressures and internal stresses, for arbitrary cross-sections, of the linked members of a chain of rigid bodies. The methods used are again geometrical but the masses of the members have to be taken into consideration. The analysis is

due largely to Wittenbauer and to Federhofer himself. A second fundamental problem is to determine the force P_Q, acting at a definite place Q in a definite direction, which must be added to given impressed forces to produce a prescribed state of acceleration of a movable system of known mass distribution.

The name of Lamé has become associated with certain functions satisfying a linear differential equation of a special type in which the coefficient of the dependent variable is a doubly periodic function. These functions are related to the ellipsoidal harmonics and include many important functions as particular cases. When the ellipsoid is replaced by an elliptic cylinder, and Laplace's equation by the equation of wave-motion, the corresponding functions are those of Mathieu. The differential equation then has a simply periodic coefficient and is in fact identical with an equation studied by Laplace in 1777 when he explained his method of successive approximations. A report on these and allied functions is particularly timely now that the applications of such functions are so numerous and it is fortunate that the report has been written by a man like Strutt who is engaged in industrial research and is familiar with many of the important applications. The report is well written and contains much new matter.

Hohenemser's report is intensely practical and will be eagerly read by men who are not satisfied until they have obtained numerical results. The problem of elastic vibrations is at once formulated with the aid of integral equations and a good sketch is given of methods of iteration for the determination of the normal vibrations. As an alternative to a successive derivation of these, there is a method of van den Dungen, by which some of them may be obtained simultaneously. Problems of elastokinetics are next formulated as variation problems and, after examples have been indicated of the use of Rayleigh's principle, an account is given of its elaboration by Ritz, Galerkin, Kryloff, and others. In this work the approximations are made by functions which do not satisfy the differential equation (or equations). An alternative method, developed recently by Bergmann, Trefftz, and Friedrichs, employs functions which satisfy the differential equation but do not satisfy (all) the supplementary conditions. Some limitations of this method are pointed out.

The second part of the report describes methods depending on the use of linear differential equations such as those of Liouville and Sturm. The method of successive approximations is explained and an approximate method of calculating the frequencies of vibration of composite systems is indicated. Some remarks are made also on the solution of frequency equations expressed in determinant form. The report closes with a discussion of the numerical results obtained in special problems relating to the torsional and transverse vibrations of rods, the critical speeds of revolving shafts, and the vibration of plates. The report condenses the results obtained by wide reading, many of the journals being inaccessible to the general reader.

Harald Bohr expounds a branch of analysis which has been created almost entirely during the last few years. The preliminary chapter on purely periodic functions is characterized by a freshness of treatment in which emphasis is laid on the uniqueness theorem, the Parseval relation, and the theorem of Fejér. The extent to which one of these theorems is a consequence of another is indicated and much use is made of an extension of Parseval's theorem which

involves the operation of folding and is important in the theory of almost periodic functions. The main theorem which is finally proved at the end of the lectures is that the class of almost periodic functions is identical with the class $H\{s(x)\}$ of functions which can be uniformly approximated by finite sums $s(x)$ composed of terms of type $a_n \exp(i\lambda_n x)$, where the coefficients a_n are complex quantities and the exponents λ_n real quantities all of which can be chosen freely. The analysis deals largely with mean value theorems, the multiplication theorem, the uniqueness theorem and its equivalence to the Parseval relation, limiting forms of integrals, and a convergence theorem for infinite series of type $s(x)$ when the Fourier exponents are linearly independent. The report closes with an account of some generalizations of the idea of an almost periodic function. The work is all of a high standard and will stand the scrutiny of mathematicians who insist on a statement of all the saving restrictions.

H. BATEMAN

KAMKE ON PROBABILITY

Einführung in die Wahrscheinlichkeitstheorie. By Erich Kamke. Leipzig, S. Hirzel, 1932. vii+182 pp.

This excellent *Theory of Probability* is an elaboration of the author's paper *Über neuere Begründungen der Wahrscheinlichkeitsrechnung.** In this paper Kamke sets forth a new foundation for probability and gives a critique of some closely related theories, those of R. von Mises, Dörge, and Tornier. Von Mises—whose work marks the beginning of a new era in probability theory—discards the conception of "equal likely events" as a proper basis for probability, in favor of a treatment based upon infinite sequences. If a die is thrown, and the results recorded,—for example, 2, 5, 3, 3, 6, 1, 4,—a sequence is obtained which may be thought of as continuing indefinitely. Now let m be the number of times that the five-spot appears in the first n throws. Some authors naively write: $\lim m/n = 1/6$. But, in general, no such limit exists. We know nothing about the future behavior of the die. A rigorous treatment of probability must divorce itself from all physical considerations. This the newer theories do. We may write: $a_1, a_2, \cdots, a_n, \cdots$, as symbols for a set of numbers, restricted to the values 1, 2, 3, 4, 5, and 6. And among all possible sequences of this description we may *choose to consider only* those for which $\lim m/n = 1/6$. And the properties of such sequences may be ascertained with mathematical rigor. Sometimes the sequence definition is called the "statistical" definition of probability. This is unfortunate, for the word "statistical" is likely to carry a reader's mind to the loose "limit" first mentioned. A sharp distinction between such a *pseudo-limit* and a real limit is fundamental for all clear thinking in probability. The theory of R. von Mises is grounded upon two axioms; that is, he *considers* only those sequences for which both axioms are valid. The first axiom relates to the existence of limits, such as has just been illustrated. The second axiom involves a selection—from the original sequence—of terms by some scheme which relates to their position or order—to the subscript r of a_r—

* Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 42 (1932), pp. 14–27.

and not to the value which a_r takes on. The chief objective of this axiom II is that such a "position-selection" should yield a new sequence with the same limit values. A periodic sequence, except in trivial cases, would not satisfy Axiom II. Kamke observes, however, that the existence of sequences conforming to the two axioms of von Mises has not been established. Kamke's theory gravitates more in the direction of that of Copeland who established existence theorems for sequences selected from an original sequence by using arithmetic progressions of the subscripts. Indeed, Kamke attributes (pp. 134, 140) to Copeland two of the theorems appearing in his book; and he attributes (p. 102) to Tornier a theorem on "probability fields,"—such a field being a countable set of sequences satisfying certain conditions.

However, Kamke's approach to the subject is not the axiomatic approach. He considers, indeed, sequences for which certain limits exist; but he does not lay down at once as axioms a complete set of statements from which all further statements are to be obtained deductively. Rather, he introduces his hypotheses as they are needed.

Although the book deals here and there with biologic or economic problems (the Marbe material on the succession of male and female births (iteration or "runs," pp. 58–60); the Mendel law of heredity (pp. 70–73); the Lexis dispersion theory (pp. 161–176)) Kamke's primary interest lies in purely mathematical problems, such as the Tschebyscheff problem (p. 110): What is the probability that a common fraction will be in its lowest terms? On page 3, Kamke begins to make preparation for this problem by laying down the common fractions in a *definite order*. Indeed, until such a *sequence* is laid down, the problem has *no solution*—a fact overlooked by many writers who have treated this problem. A number of game problems, too, appear as illustrations.

The book consists of two parts of about equal length. The first part deals with probability-sequences, singly or in finite number; and the second part with probability-fields, countable sets of probability-sequences. A probability-sequence is one in connection with which certain limits exist. Part I deals with two addition laws, a division law, and five multiplication laws. An exact mathematical definition of independence is used, following the lead of Hausdorff. A treatment is given of Newton's formula for repeated trials, mean value, variance, risk, and the case where an infinite set of distinguishing marks appears among the terms of the sequence. Part II deals with the existence of probability-fields with a prescribed matrix of distinguishing marks. For these marks, probabilities must exist as limits in the individual sequences, and one or the other of two multiplication laws must be valid. For these fields, then, certain multiplication laws are proved involving upper and lower limits. A Bernoulli probability-field is one satisfying the more restrictive multiplication law, and having a *constant* probability for the appearance of a specified mark in each individual sequence. The law of Bernoulli is proved, and the approximation formulas of Stirling, Laplace, and Poisson. By starting in turn with each of the first k terms of a sequence, and annexing every kth term thereafter, there is formed from a given sequence k partial sequences. If for each positive integral k, the sequence thus formed is a Bernoulli field—so far as it goes—with constant probability, the original sequence is called a Bernoulli sequence. Not every probability sequence is a Bernoulli sequence, but for a

given probability w, with $0<w<1$, a Bernoulli sequence can be constructed; in fact, for the same w, a set of sequences having the power of the continuum. The "law of large numbers" and the "fundamental law of probability" leading to the "normal" probability function and its integral are then established. The Lexis dispersion theory concludes the text proper. Then follow: a one-page 4-place probability integral table; answers to the 18 exercises given in the text; a short list of important books and papers; an index for authors and subjects.

Kamke's book is by no means an elementary text. Though it presupposes no knowledge of conventional probability material, the reader should be well grounded in analysis. Kamke regards probability, not as a mysterious field closely allied to philosophy, but simply as a branch of function-theory directed toward infinite sequences of designated types. The book is compact; but it is extremely well written, and it is also surprisingly free from typographical imperfections. In the opinion of the reviewer, this is one of the most distinctive and important treatises on probability that have appeared in recent years.

E. L. DODD

DUBISLAV ON FOUNDATIONS

Die Philosophie der Mathematik in der Gegenwart. By Walter Dubislav. (Philosophische Forschungsberichte, Heft 13.) Berlin, Junker und Dünnhaupt Verlag, 1932. vi+88 pp. R.M. 3.80.

Die Definition. By Walter Dubislav. Third, revised and augmented edition. (Beihefte der *Erkenntnis*, Heft 1.) Leipzig, Felix Meiner, 1931. viii+160 pp. R.M. 14.00.

In the little book *Die Philosophie der Mathematik in der Gegenwart*, Dubislav gives an incisive discussion of certain problems of the foundations of mathematics. The problems, "What is mathematics?" and "What can it claim to prove?" are subjects of vigorous controversy; our author gives the word to all contenders, and then has his own wise comments on the views of each.

After a presentation of elements of symbolic logic—a tool essential to further progress—he devotes a section to the "metamathematical" group of problems. The very fact that the word is used indicates Dubislav's place in the camp of Hilbert. As Ramsey puts it,* metamathematics consists of real, meaningful assertions about mathematics, which is itself meaningless; and, precisely, the main problems—those which this book deals with—are those of consistency, of the possibility of decision (general validity and fulfillability), and of completeness. The treatment in a work of this size must, of course, be fragmentary, and Dubislav chooses to emphasize the latest results obtained by the Hilbertians—for instance, Gödel's proof that all logico-mathematical calculi which comprise arithmetic contain indeterminable statements, one of which asserts the self-consistency of the system,† and Dubislav's own contribution to the problem of decision.

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* *The Foundations of Mathematics*, p. 68.

† Kurt Gödel, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme*, Monatshefte für Mathematik und Physik, vol. 38 (1931), p. 173.

The most valuable part of the final section is a clear statement of the views of six contending schools on the nature of mathematics. Dubislav's comment on these various doctrines is both penetrating and, in the reviewer's opinion, correct. The first theory, that most vigorously opposed, is the older, Kantian intuitionism (as it is named in the chapter on the subject-matter of mathematics); it is closely allied to criticism (Kritizismus) (the name used in the discussion of the foundations of the science). The Kantian doctrine holds that mathematics contains, over and above logic, synthetic judgments whose truth is guaranteed by their obviousness to every reasonable intellect. It must be said that the appeal to unanimity among sane minds, even if that unanimity could be ascertained, is an appeal to the principle of induction, itself an even less obvious principle than those which were so fundamental for Kant. "Criticism amounts to the assumption of Hilbertian metamathematics, supplemented by Kantian teachings on the nature of mathematics. For the adherents of criticism this adjunction is an essential deepening of Hilbert's conceptions; for the strict formalist an illegitimate adulteration of mathematics with partly mystical, partly metaphysical trains of thought."

If Kant held that man knows more things intuitively than most of us would now admit, Brouwer, also appealing to intuition, is so parsimonious as to the data which he trusts that admissible mathematics is tremendously complicated, while a large part of what we have called mathematics is denied all justification. That, of course, is no ground for rejecting it; more serious is the fact that Brouwer "is not in a position precisely to characterize those constructions—supposed to be independent of every language—of which mathematics is said to be built."

Other theories are: empiricism—"mathematics is that natural science which seeks to investigate the more general properties of objects"; conventionalism—"it is neither a natural science nor identifiable with logic, but a grand system of self-consistent demands, with their consequences"; logicism—"mathematics, as a branch of logic, is a system of tautological, unconditionally true statements."

All these theories are rejected—though with cordial recognition of their valuable contributions—in favor of formalism, for which mathematics is but a game with certain marks and rules. Logic is a part of this game, but a comparatively elementary one, for it lacks the infinity and selection axioms. The poverty of mathematics thus defined, its lack of contact with any reality, finds ample compensation in the wealth of real things which can replace the marks. "It is probable that formalism alone can succeed in bringing pure logic and pure mathematics to the state [freedom from 'real controversy'] described by Gauss. Hereto it is peculiarly adapted; for on its soil, free from metaphysics and mysticism, we can pursue logic and mathematics, with Fourier because of their great services to science, or with Jacobi pour l'honneur de l'esprit humain." Yet even formalism, it seems to us, can not dispense with an intuitional basis. That the substitution of b for x always changes $(a-x)$ into $(a-b)$ may seem far more certain than the principle of causality; but, if 99% of our acquaintances believed in a magical metamorphosis in connection with substitution, would our confidence remain?

The book concludes with two briefer chapters. One treats of the infinite,

dwelling on the relations between Brouwer's and Hilbert's finitisms; the other tells how the various doctrines regard the relation of mathematics to the real world.

The second book to be considered, *Die Definition*, is quite as important a contribution to science—and is, naturally, broader in scope, as the definition of definition is not an affair of mathematics exclusively. The first, briefer, of the two sections of the book, reviews the main concepts of definition which have been proposed. There is the Aristotelian definition—determination of the essence of a thing (Wesensbestimmung). There is the Kantian definition—the determination of a concept (Begriffsbestimmung). A definition may, thirdly, be the report (Feststellung, nicht Festsetzung) of the meaning or use of a sign; this conception of the definition most closely approximates that of the dictionary. The fourth type of definition is the establishment (Festsetzung, nicht Feststellung) of the meaning or use of a sign. The first section closes with a brief history of its development. We find but one thing to criticize in this summary of definitions—that it is almost wholly devoid of those examples which so greatly clarify the use of highly abstract concepts.

The theory of definition most pleasing to Dubislav is an outgrowth of the Frege-Peano theory. Within a given discipline a definition is simply a rule for substitution. The method of such substitutions can, however, not be used without precautions—precautions, which, according to Dubislav, should be these: —that a definition be eliminable, and such that the repeated use of the substitution which it expresses in correct statements always leads to correct statements. A chief virtue of the Frege theory is the sharp distinction between the assertion and its clothing in language; its failings are the lack of clarity as to the permissible connections of signs in the definitions, and the confidence placed in "self-evident" axioms.

It is the formalistic "game theory" which gives a much more satisfactory account of the "Festsetzung" type of definition; it fills the gap in Frege's theory, giving an exact characterization of the external structure of a formula. In this section, the definition is regarded as the substitution of a mark for another collection of marks; since the main difficulty consists in the certain avoidance of paradoxes, there is an exposition of Russell's hierarchy.

We can not, however, neglect the "Feststellung" definition; for natural objects, as well as symbols, are defined. In this sense a definition coordinates sign with object. The dependence of "truth" on language, the preponderant importance of the negative result of an experiment, Hertz' analysis of a scientific theory (object→sign→sign→object), the approximate, statistical character of the verification of a law, are all needed as preliminaries to the discussion of the definition of an object. This second type of definition is described in detail, and illustrated by the definitions of order on a circle and of measurement on a line.

"Definition," as we saw at the outset, has yet other meanings. These, which have, in general, less to do with mathematics, are thoroughly treated by Dubislav, who ends his book with a clear summary of all types.

Dubislav is a master juggler of language. He can construct sentences whose analysis might grace an honors examination in grammar. The last three-quarters of a sentence on page 35 of the *Philosophie* reads: "Eine Definition

durch Abstraktion besteht darin, dass man versucht, und zwar, ohne eine quasi-metaphysische Hilfsannahme zu machen, vergeblich versucht, aus einer Definitionsgleichung der Form $(fx=fy)=(xRy)Df$, wobei R eine transitive und symmetrische Relation ist, durch die ersichtlich das Zeichen $f\hat{z}$ nur im Zusammenhang mit anderen eingeführt wird, das betreffende Zeichen $f\hat{z}$ dennoch als ein vollständiges zu gewinnen, dem dann der durch die betreffende Definition angeblich in Gestalt eines idealen Objecktes schöpferisch erzeugte Gegenstand entsprechen soll."

In spite of the unusual powers revealed in this quotation, Dubislav rarely indulges in such telescopic delights. He writes with a firm, accurate style, revealing a mind of refreshing vigor.

In treatises on such abstract topics, one smiles, noting the warmth and enthusiasm with which the author defends his cold, unemotional concept of truth, and attacks the psychologists and metaphysicians who wish to be less restricted. "A purported statement about reality, based neither immediately nor indirectly on observations or their results, is a chimaera." "A science is no collection of sermons of salvation or messages of deliverance or rationalized myths or intellectual prose poems, based on emotional projections and their pseudo-rationalizations; it is, rather, a collection of statements which can, at least in principle, be tested by experience, together with the appropriate observations, experiments, and calculational transformations." Mathematical and philosophical thought have been definitely enriched by these contributions to their foundations.

E. S. ALLEN

SHORTER NOTICES

Mathematical Tables. Volume 2. Prepared by the Committee for the Calculation of Mathematical Tables. London, British Association for the Advancement of Science, 1932. viii+34 pp.

The first volume of the British Association Mathematical Tables was described in this Bulletin, vol. 38, pp. 458–462. The second volume is devoted entirely to the tabulation of certain Emden functions. The origin of the work is explained in the preface: "In 1930 Sir Arthur Eddington, as President of the Commission for the Constitution of the Stars of the International Astronomical Union, suggested to the Committee for the Calculation of Mathematical Tables that they should undertake the tabulation of solutions of Emden's equation, which is fundamental in modern theories of the internal structure of the stars. The Committee, which is anxious to establish itself as a clearing-house for tables of this nature, was authorised by the Association to put the necessary work in hand."

Emden's equation may be written

$$\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}+y^n=0.$$

The solutions here tabulated are those which are regular at $x=0$ and are defined by the initial conditions: $x=0$, $y=1$, $y'=0$. For $n<5$ these solutions all vanish for some finite positive value $x=x_0$, the value of x_0 increasing with that of n, as shown in the following table.

n	x_0	n	x_0	n	x_0
1.0	3.1415927 $(=\pi)$	2.5	5.3552754	4.0	14.9715476
1.5	3.6537537	3.0	6.8968486	4.5	31.8364636
2.0	4.3528746	3.5	9.5358054	5.0	∞

The tables are constructed for the values of n indicated above and, with the exception of $n=3$ and $n=5$, are carried as far as the first root $x=x_0$, the interval for x being usually 0.1. In the case of $n=3$ the table extends to the third root, which is slightly greater than 102. For $n=5$ the table extends as far as $x=5$. In two cases, $n=1$ and $n=5$, the equation admits of integration in terms of simple functions and the tabulation has been abbreviated by the omission of certain auxiliary functions.

In addition to the tabulation of the fundamental solution (y) and of its derivative (y'), higher derivatives have been given to facilitate interpolation by Taylor's theorem, and eight auxiliary functions, required in astronomical calculations, are also tabulated.

The introduction and explanation of the tables was written by Mr. D. H. Sadler of H. M. Nautical Almanac Office. The cost of preparing the tables was borne by the British Association and the cost of printing this volume by the International Astronomical Union.

W. R. LONGLEY

Elements of the Theory of Integers. Second Edition. By Joseph Bowden. Published by the author, Garden City, New York, 1931. 268 pp.

The changes from the first (1903) edition consist mainly in the correction of errors and a revision of the chapter on congruences. The author gives an axiomatic treatment of the theory of rational integers as far as the theorem of unique decomposition into prime factors. The final chapter on congruences ends with the theorem of Fermat for a prime modulus. The extremely limited subject matter makes the book hardly acceptable for a student of the theory of numbers. For a student interested in the axiomatics of arithmetic the book is open to several criticisms. A general treatise on the axiomatic structure of algebraic systems would be certainly more fascinating and scarcely more difficult. No discussion of the independence and consistency of the system of axioms is given. The author uses simplified spelling and an original symbolism.

H. T. ENGSTROM

The Law of Gravitation in Relativity. By H. C. Levinson and E. B. Zeisler. University of Chicago Press, 1931. 127 pp.

This book, in addition to being an exposition of tensor analysis, attempts to formulate all possible laws of gravitation which are obtained from a set of linear differential equations of the second order, in which the dependent variables are the components of the metrical tensor. In addition to Einstein's assumption the authors consider the case where space time is conformal to a flat space (vanishing of the Weyl tensor) and carry out the details of the calculation of a centro-symmetric field under this hypothesis. Another possibility where the vanishing tensor carries six labels (two from the metrical tensor and four from the Weyl tensor) is mentioned, but the details are not worked out nor is the geometrical significance of this tensor obtained. The authors have quite evidently worked through the whole theory independently, and the book makes interesting reading, provided one is already somewhat familiar with the methods of tensor analysis.

F. D. MURNAGHAN

Critique of Physics. By L. L. Whyte. New York, W. W. Norton, 1931. xi+196 pp.

This book is more philosophical than mathematical and is mainly an analysis and criticism of the postulates and concepts underlying relativity theory and quantum mechanics. The author believes that the fundamental role of Planck's constant h is to furnish in combination with other atomic constants a natural unit of length in terms of which sizes of atoms and of crystal lattice structures, etc., may be expressed. The book is more a program for research than an exposition of a new theory, and it is the hope of the writer that the publication of such a program may stimulate the development of a new theory. The book is not at all like the multitude of popular controversial books dealing with the theory of relativity, and the writer has evidently thought seriously about the topics he discusses. There are two short and interesting historical appendices dealing with the theory of relativity and with the quantum theory.

F. D. MURNAGHAN

Where Is Science Going? By Max Planck. Prologue by Albert Einstein. Translation and Biographical Note by James Murphy. New York, W. W. Norton and Co., 1932. 221 pp. Price, $2.75.

A great scientist and great man here analyzes the present state of physics and a half century of its development with reference to its implications for the further progress of science and for certain major interests of human welfare in general. The prologue by Einstein, discussing the motives which impel some of our choicer spirits to scientific research, furnishes a fitting prelude to the general theme of the book. The introduction by Murphy will stimulate the reader's interest in Planck both as scientist and as man. Chapter I, *Fifty years of science*, (pp. 41–63), gives a brief non-technical but illuminating presentation of the background of the present state of physics. The second chapter argues for the reality of the external world and sharply opposes a bare positivistic philosophy of science. This reality can never be fully attained; but "it is not the possession of truth, but the success which attends the seeking after it, that enriches the seeker and brings happiness to him." The third chapter is devoted to the scientist's picture of the physical universe. The next two chapters treat of causation and the freedom of the will. It is too much to expect any one to deal satisfactorily with the latter problem. With respect to the former the considered judgment of Planck is that the principle of causality should be given such a modified formulation as to restore it to strict validity. The final chapter deals with the passage from the relative to the absolute. The book closes with an epilogue in the form of a Socratic dialogue by Planck, Einstein and Murphy. Throughout this stimulating volume the treatment is as free of technicalities as the nature of the subject admits.

R. D. CARMICHAEL

Kurventheorie. By Karl Menger. Herausgegeben unter Mitarbeit von Georg Nöbeling. Berlin and Leipzig, B. G. Teubner, 1932. vi+370 pp.

In the last chapter of Pierpont's *Theory of Functions of Real Variables*, vol. II (1912), there is a discussion of several geometric concepts, including that of a plane curve. Various definitions which might appear reasonable at first sight are examined and shown to be inadequate, with the net result that the question, "What is a curve," is not answered. After the lapse of twenty years, we have a book devoted to this question, which gives the answer and discusses extensively the properties of curves, in a way little dreamed of in 1912.

The book under discussion is to be the second volume of a work entitled *Mengentheoretische Geometrie.* In addition to being the first treatise on the theory of curves and their topological properties, the book performs the further service of gathering together in an organic whole much of the research work on this subject done since the war—work which, from the nature of its piecemeal appearance, must have seemed to many to be of a pointless character. To Americans this book is of special interest, in view of the generous acknowledgement of their contributions to the subject in the publishers' announcement and in the bibliographical notes.

In the first chapter are given elementary notions from point-set theory for later use, and brief discussions of various types of point sets, such as irreducible

continua, locally connected continua, simple arcs, ordinary curves, and Cantorian curves—all of which are shown to be too narrow or too broad to serve as a definition of the term *curve*. Chapters II and III contain the definition of a curve essentially as a continuum one-dimensional at each point, and theorems on the structure of the sub-sets of points of various orders. In Chapters IV, V, and VI we have theorems on sums of curves, covering theorems, separation theorems, the n-Beinsatz, and the n-Bogensatz. The remaining six chapters deal with hereditary local connectivity, regular curves, rational curves, trees, cyclic continua and cyclic elements, and the universal curve. Finally we have a summary of the principal theorems, mention of topics not treated, and a brief discussion of certain unsolved problems.

Since the name of the author guarantees the excellence of this book, the reviewer feels that he can do little more than commend it to students of topology as indispensible and to prospective writers of mathematical treatises as a model of exposition, and notice certain special features. One of these is that, since the curve theory is in some aspects merely a discussion of 0-dimensional and 1-dimensional sets, the book will be of assistance to students reading the corresponding more general work in the *Dimensionstheorie* for the first time. The *Beispiele*, which include not only *examples* in the usual sense but also related bits of theory, and the references at the end of each section, are obviously of great value. Although, as the author points out, there is no essential loss of generality in supposing curves to be imbedded in euclidean space, the reviewer feels that in some cases proofs would appear less cumbersome if the common practice of regarding the curve itself as a compact metric space were followed. There are fewer instances where the continuity of thought is broken by the insertion of lemmas in the proofs of theorems than in the *Dimensionstheorie*. In one or two cases it appears that present or future workers ought to be able to find simpler demonstrations. (See, for example, L. Zippin, Annals of Mathematics, vol. 34, p. 95.) The choice of material in such a book as this is largely governed by physical limitations and the author's own work; it may, however, be worth while to note the small space allotted to linear graphs, external properties of curves, and the structure and decomposition of irregular curves. No systematic search for slips in reasoning and misprints has been made, but a certain amount of sampling indicates that these can be but few in number. Last, but not least, there is an excellent index and it is hoped that this will be a feature of any revision of the *Dimensionstheorie*.

In conclusion it may be remarked that, although it is often said that the appearance of a book on a subject indicates that its life as a field of research is nearly ended, there are reasons for thinking that this book may be a stimulus to further discoveries, both because of its rendering much of the work of the past accessible as a whole and because of the unexplored territory indicated by the author himself.

W. A. Wilson

NOTES

A portion of the library of Oliver Heaviside, who died in 1925, is now on exhibition at the Massachusetts Institute of Technology. The collection includes a number of autograph letters and other memorabilia gathered by the late Dr. B. A. Behrend. It will shortly be sent to England to be preserved in the library of the Institution of Electrical Engineers.

The Princeton University Press will publish sometime during the early summer a book by Dean L. P. Eisenhart on *Continuous Groups*.

A translation of Enriques' *Lezioni di Geometria Proiettiva* has been made by Professor H. R. Phalen, and published by him, reproduced by a duplicating process in a limited edition. Orders should be sent to Professor Phalen at St. Stephen's College, Annandale-on-Hudson, N. Y.; the price is $3.75.

The Royal Academy of Bologna announces the following subjects for its Adolfo Merlani prize in mathematics, for which competition closes December 31, 1934: (1) *A contribution to the study of the roots of the integrals of linear differential equations (in the real domain)*; (2) *A contribution to some question in the mechanics of fluids, preferably in aerodynamics.* Competing memoirs must be written in Italian, and may be in manuscript or already in print.

The Physico-Mathematical Society of Kazan announces that the seventh award of its Lobatchefski prize will be made on February 26, 1934. Competition closed February 1, 1933.

Cambridge University announces the following awards: Smith's prizes to E. A. Maxwell, of Queen's College, for an essay entitled *The invariants of certain surfaces*, and to R. H. Stoy, of Gonville and Caius College, for an essay entitled *The planetary nebulae*; Rayleigh prizes to W. E. Candler, of Trinity College, for an essay entitled *The stability of the rings of Saturn*, to C. Strachan, of Corpus Christi College, for an essay entitled *Reflection by monomolecular films*, and to M. H. H. Walters, of King's College, for an essay entitled *The effect of stellar encounters on the orbits of binary stars.* The following subject is announced for the Adams prize for the period 1933–34: *The mathematical representation of unsteady flow in fluids.* Competition for this prize is open to all persons who have at any time been admitted to a degree in Cambridge University.

Sir Richard Glazebrook, formerly chairman of the Aeronautical Research Committee, has been awarded the gold medal of the Royal Aeronautical Society.

The Royal Society of Edinburgh has awarded its Makdougall-Brisbane prize for 1930–32 to Dr. A. C. Aitken, of the University of Edinburgh, for various contributions to mathematics published in the Proceedings of that Society.

Dr. Sydney Chapman, chief professor of mathematics at the Imperial College, South Kensington, has been elected president for 1933 of the Royal Meterological Society.

Dr. William Bowie, chief of the division of geodesy of the United States Coast and Geodetic Survey, has been reelected honorary president of the Pan American Institute of Geography and History.

Dr. C. F. Roos, formerly permanent secretary of the American Association for the Advancement of Science, has been awarded a Guggenheim fellowship to aid in writing a book on dynamical economics in consultation with European scholars.

Announcement of the establishment of the Elihu Thomson professorship of electrical engineering at the Massachusetts Institute of Technology was made on March 29 by President K. T. Compton, at the dinner given at the Institute in honor of Professor Thomson's eightieth birthday.

The Council of the Royal Society of London has recommended Professor J. H. M. Wedderburn, of Princeton University, for membership in that Society.

Professor Albert Einstein has resigned his membership in the Prussian Academy of Sciences. He has accepted professorial appointments at the Universities of Madrid and Paris, in addition to his professorship at the Institute for Advanced Study, Princeton.

Dr. Gerhard Haenzel has been appointed professor of geometry at the Karlsruhe Technical School.

Professor J. Proudman, of the University of Liverpool, has been transferred from the professorship of applied mathematics to that of oceanography.

Professors Lipót Fejér and Tullio Levi-Civita will arrive in this country several weeks before the Society's meeting at Chicago in June, at which they are both to lecture. Professor Fejér will lecture during April and May at Brown University, Harvard, Columbia, Princeton, the University of Pennsylvania, Cornell, and the University of Cincinnati, and before a sectional meeting of the Mathematical Association of America at the University of Virginia; his topics will be *On new properties of the arithmetical means of the partial sums of Fourier series,* and *On the characterization of some remarkable systems of points of interpolation by means of conjugate points, of the Cotes' numbers, and of certain extremal properties.* His lecture at the Chicago meeting will be *On the infinite sequences arising in the theories of harmonic analysis, of interpolation, and of mechanical quadratures.* Professor Levi-Civita will lecture at Princeton, Harvard, and Brown during the last week in May and the first week in June, on *Secular effects of tides on the motion of a planetary system* and on *Adiabatic invariants.* His lectures at Chicago will be *On some mathematical aspects of the new mechanics* and *Nets on a surface and extension of trigonometry.*

Professor T. C. Esty, of the department of mathematics at Amherst College, has been made vice-president of the College.

Professor R. H. Fowler will be in residence at the University of California for the second semester of the present academic year.

Associate Professor W. C. Graustein, of Harvard University, has been promoted to a professorship of mathematics.

Associate Professor Einar Hille, of Princeton University, has been appointed professor of mathematics at Yale University.

Associate Professor M. H. Stone, of Yale University, has been appointed associate professor of mathematics at Harvard University.

The following appointment to an instructorship in mathematics is announced:
Cornell University, Dr. A. H. Black.

Jules Andrade, correspondent of the Paris Academy of Sciences in the section of mechanics, known for his work in chronometry, died February 25, 1933, at the age of seventy-five.

Dr. Felix Auerbach, professor of theoretical physics at the University of Jena, and his wife committed suicide on March 1, 1933. Professor Auerbach was seventy-seven years old.

Professor Eugenio Bertini, of the University of Pisa, died February 23, 1933, at the age of eighty-six.

Professor Alfred Haar, of the department of mathematics at the University of Szeged, Hungary, died March 16, 1933, at the age of forty-eight.

The death is announced of Professor C. Köhler, of the University of Heidelberg, at the age of fifty-three.

Joseph Kürschák, professor of mathematics in the Polytechnical Institute of Budapest, died in March, 1933, at the age of sixty-nine.

Professor Augustin Mesnager, of the Paris Ecole des Ponts et Chaussées and Conservatoire des Arts et Métiers, member of the Paris Academy of Sciences in the section of mechanics, died recently.

Enrico D'Ovidio, professor emeritus of mathematics at the University of Turin, died March 21, 1933.

Dr. Karl Rodenberg, professor of descriptive geometry at the Hannover Technical School, is dead.

The death is announced of Professor D. Seliwanoff, of the University of Prague, at the age of seventy-seven.

Professor Magnus de Sparre, of the Catholic University of Lyons, correspondent of the Paris Academy of Sciences in the section of mechanics, has died at the age of eighty-three.

Dr. F. Gomes Teixeira, of the Academy of Sciences of Coimbra, died at Oporto at the age of eighty-two.

Dr. E. W. Hobson, emeritus professor of pure mathematics at Cambridge University, died April 18, 1933, at the age of seventy-six.

Dr. R. E. A. C. Paley, the distinguished young English mathematician, International Research Fellow at the Massachusetts Institute of Technology, was killed by an avalanche while skiing near Banff, Alberta, on April 7, 1933, at the age of twenty-five.

L. M. Defoe, professor emeritus of mechanics at the University of Missouri, died April 3, 1933, at the age of seventy-two. Professor Defoe had been a member of the American Mathematical Society since 1900.

Professor F. B. Williams, of Clark University, died March 7, 1933. Professor Williams had been a member of the American Mathematical Society since 1914.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THIS SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

123. Professor G. A. Bliss and Dr. M. R. Hestenes: *A note on the problem of Bolza in the calculus of variations.*

In this paper it is shown by a simple reapplication of the first necessary condition on a minimizing arc for the problem of Bolza that for every such arc there exists a function $F=\lambda_0 f+\lambda_\alpha \phi_\alpha$ for which $F-y_i' F_{y_i'}$ and F_x satisfy a Du Bois Reymond equation similar to those satisfied by the functions $F_{y_i'}$ and F_{y_i}. These equations are dependent for a minimizing arc which is normal and non-singular. For other arcs no proof has as yet been given that this is so. Consequences of these results are given. It is shown that for a normal minimizing arc the continuity of $F-y_i' F_{y_i'}$ is a consequence of the continuity of $F_{y_i'}$ and the necessary condition of Weierstrass. (Received March 22, 1933.)

124. Dr. M. R. Hestenes: *Sufficient conditions for the problem of Lagrange in the calculus of variations.*

In this paper a set of sufficient conditions for the problem of Lagrange with fixed end points is given in which it is assumed that the arc E_{12} under consideration is normal on the interval x_1x_2 but no assumption is made regarding normality on the sub-intervals of E_{12}. This result is attained by replacing the usual condition that E_{12} contains no point conjugate to the point 1 by a new condition which states that a certain quadratic form is positive on the interval x_1x_2 and positive definite at one value of x on x_1x_2. This new condition is a consequence of the theory of broken extremals applied to the problem of the second variation. It is closely related to a condition given by Bliss for variable end point problems in the plane and also the fundamental quadratic form of Morse. The relations between the sufficient conditions here given and known sufficient conditions are discussed. In particular an example is given in which the minimizing arc E_{12} satisfies the conditions of this paper but which is not normal on every sub-interval x_1x_3 nor on every subinterval x_2x_2 and hence does not satisfy the sufficient conditions given heretofore. (Received April 8, 1933.)

125. Professor Oswald Veblen: *Geometry of two-component spinors.*

This paper shows how a theory of two-component spinors can be derived from the geometry of the light-cones in the tangent spaces of the space-time

manifold of general relativity. It is to be published in the Proceedings of the National Academy of Sciences for April, 1933. (Received April 15, 1933.)

126. Professor R. E. Langer: *An inverse problem in differential equations.*

In an investigation (by Professor L. B. Slichter of the Massachusetts Institute of Technology) of the electrical resistivity of the earth's crust at inaccessible depths, an electrode is made to supply a current to the earth at its surface. The electrical potentials resulting *at the surface* in the surrounding region are observable, and constitute the sole obtainable data. From them the conductivity of the earth as a function of the depth is to be computed. The problem leads to the following formulation. In the equation $\{d(\sigma(x)dy/dx)\}/dx - \lambda^2\sigma(x)y=0$, λ is a parameter, $0<\lambda<\infty$, and $\sigma(x)$ a positive, differentiable, but unknown function. If $y_1(x, \lambda)$ is the solution of the equation which vanishes at $x=\infty$, then $k(\lambda)\equiv[y_1(x, \lambda)/\{d(y_1(x, \lambda))/dx\}]_{x=0}$ is a known function of λ. From it $\sigma(x)$ is to be computed. In order to do this, the equation is transformed into $d(\log \sigma(x))/dx = \{v'(x, \lambda)+1\}/\{v(x, \lambda)\} - \lambda^2 v(x, \lambda)$. If one sets $v(x, \lambda) = -1(\sum_{j=0}^{\infty} a_j(x)/\lambda^j)/\lambda$, and substitutes this in the equation, the condition that the result be independent of λ, together with $v(0, \lambda)\equiv k(\lambda)$, yields a system of recurrence relations from which the Taylor's series for $\sigma(x)$ may be computed. (Received March 22, 1933.)

127. Professor I. S. Sokolnikoff and Dr. E. S. Sokolnikoff: *On a resolution of linear differential systems.*

A problem of resolving a linear differential system into a product of two or more systems of lower order is considered in this paper. The fact that such a resolution is not always possible, even when the given differential operator is factorable, is obvious. This paper furnishes the necessary and sufficient conditions for resolving the system into two equivalent systems of lower order when the given differential operator is decomposable into two or more factors. The results are specialized for the case of a differential system of second order, and some examples of resolution of differential systems are given. (Received March 21, 1933.)

128. Dr. H. P. Thielman: *On the invariance of a generalized Gramian under the group of linear functional transformations of the third kind.*

In this paper a Riemannian function space is considered (A. D. Michal, American Journal of Mathematics, vol. 50, p. 516) in which the inner product of two vectors $y_1(s)$, $y_2(s)$ is defined in the following way: $(y_1y_2)_g = \int_a^b\int_a^b g(\alpha, \beta) \cdot y_1(\alpha)y_2(\beta)d\alpha d\beta + \int_a^b g(\alpha)y_1(\alpha)y_2(\alpha)d\alpha$, $g(\alpha, \beta)=g(\beta, \alpha)$, $g(\alpha)\neq 0$ in (a, b), and $(yy)_g>0$ for all real, continuous functions which do not vanish identically in (a, b). In such a space the volume of a parallelepiped made by the vectors $y_1(s)$, $y_2(s)$, $\cdots$, $y_n(s)$ having a common origin is defined by means of the determinant $|(y_iy_j)_g|$ whose general element is $(y_iy_j)_g (i, j=1, 2, \cdots, n)$. The following theorem is then proved: *A necessary and sufficient condition that the generalized Gramian* $|(y_iy_j)_g|$ $(i, j=1, 2, \cdots, n)$ *be invariant under the group of*

linear functional transformations: $y(s)=K(s)\bar{y}(s)+\int_a^b K(s,\ \alpha)\bar{y}(\alpha)d\alpha,\ K(s)\neq 0$ *in* (a, b), *is that the group be the group of motion, i.e. that it leave* $(y_iy_j)_a$ *invariant.* If $g(\alpha,\ \beta)\equiv 0$, $g(\alpha)\equiv 1$, a well known result is obtained (I. A. Barnett, Annals of Mathematics, vol. 25, p. 220, T. S. Peterson, American Journal of Mathematics, vol. 51, p. 427). (Received March 22, 1933.)

129. Professor Theodore Bennett: *Note on* (1, 2) *algebraic correspondences.*

If in a (1, 2) algebraic correspondence between the points of two planes a point P corresponds to a single point P' and a point P' corresponds to two points P_1, P_2, then the locus of points P' for which the corresponding points P_1, P_2 coincide is called the *branch locus.* It has been stated repeatedly in the literature that the necessary and sufficient condition that the image points P_1, P_2 of a variable point P' describe separate loci is that the locus of P' be tangent to the branch locus at every common point except the fundamental points. In this note an example is constructed which proves that this condition is not sufficient. Similar examples can be constructed in projective spaces of any number of dimensions. (Received March 22, 1933.)

130. Professor Arnold Emch: *On an involutorial Cremona transformation in* S_r.

In the study of the symmetric involutorial $(n,\ n)$-correspondence and its application to problems of closure an interesting problem concerning an involutorial Cremona transformation in S_r presented itself. Let r hypercones of order two be given in S_r. Now r generic tangent hyperplanes, one for each hypercone, intersect in a point $P(\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_r)$, in which the λ's are the parameters associated with these cones and tangent hyperplanes. From P, r other hyperplanes with the parameters λ_1', λ_2', $\cdots$, λ_r' may be drawn. Thus the set (λ') is involutorially connected with the set (λ). The object of the paper is to investigate this involution, with particular reference to the cases $r=2$, and $r=3$. (Received March 24, 1933.)

131. Professor Arnold Emch: *Symmetric monoids and cones of higher order in* S_3.

If $\phi_1=\Sigma x_i$, $\phi_2=\Sigma x_ix_k$, $\phi_3=\Sigma x_ix_jx_k$, $\phi_4=x_1x_2x_3x_4$, then $\Sigma k_i\phi_1^{m_1}\cdot\phi_2^{2m_2}\cdot\phi_3^{2m_3}\cdot\phi_4^{4m_4}=0, m_1+2m_2+3m_3+4m_4=n$, represents a symmetric n-ic in S_3. The number of linearly available constants depends upon the solution of the indicated partition problem in number theory. The author's paper is a continuation of previous studies of geometric forms which are invariant in symmetric substitution groups. The immediate objects are the symmetric monoids and the symmetric cones. As an example may be mentioned a pencil of symmetric sextic cones which has the property, that *every proper quadric of a symmetric pencil cuts every sextic cone of the symmetric pencil of cones in a space curve which breaks up into two space cubics.* (Received March 24, 1933.)

132. Mr. J. R. Mayor: *A generalization of the Steiner and Veronese surfaces.*

The linear system of all hyperquadrics in S_r enables one to map the points of S_r upon a locus $G_r^{\mathcal{R}}$ in S_R, $R=r(r+3)/2$, such that the hyperplanes of S_R correspond to the hyperquadrics of S_r. Any locus of dimension r in a space S_k, $k<R$, on which the points of S_r are mapped by quadratic forms may be obtained as a projection of the variety $G_r^{\mathcal{R}}$ in S_R. For $r=2$, $G_r^{\mathcal{R}}$ is the Veronese surface in S_5. The projection of $G_r^{\mathcal{R}}$ upon an S_{r+1} from a general $S_{(r^2+r-6)/2}$ which does not meet it may be considered as a generalization of the Steiner surface. By making a special choice of the space of projection, a hypersurface is obtained in S_{r+1} upon which the points of S_r are mapped by the squares of linear forms; the properties of this hypersurface are more closely analogous to those of the Steiner surface. (Received March 22, 1933.)

133. Professor V. G. Grove: *Contributions to the theory of transformations of nets in a space S_n.*

In this paper, the transformations C and E in a euclidean space of $n\geqq 3$ dimensions are studied. The relation between a congruence and the curves of a C net is studied. In particular if the lines of a congruence are perpendicular to the tangents of one (only) of the families of curves of the net N in which the developables of the congruence intersect a surface, the lines of the congruence enjoy the same property with respect to any E transform of the net. A transformation E, which is also radial, is a transformation by reciprocal radii and conversely. The theorems developed are independent of the dimension of the space in which the configurations are immersed. (Received March 9, 1933.)

134. Dr. C. B. Morrey, Jr. (National Research Fellow): *The topology of (path) surfaces.*

A hemicactoid shall be a continuous curve in 3-space consisting of a plane continuous curve without complementary domains plus a finite or denumerable set of cactoids each having exactly one point in common with the plane set and no other points in common with the other cactoids; the simple closed surfaces involved may each be taken analytic except for two vertices. It is shown that: (1) given any hemicactoid, H, and a continuous vector function, $X(U)$, defined thereon, there exists a continuous (vector) function, $U(u)$, carrying a Jordan region, $\bar{r}$, into H, such that the set of points of $\bar{r}$ corresponding to each point of H is a continuum; (2) to each such function corresponds a surface, $x=x(u)=X[U(u)]$, all these surfaces being identical (Frêchet)—thus we may represent surfaces on hemicactoids; (3) every surface can be represented on some hemicactoid, no continuum being carried into a point. Finally (a) the relations between different representations of the same surface are discussed; (b) a criterion, in terms of its given representation on a Jordan region, that a surface be non-degenerate (author's abstract, January 1933) is developed; and (c) every saddle surface bounded by a Jordan curve is shown to be non-degenerate. (Received March 24, 1933.)

135. Dr. C. B. Morrey, Jr. (National Research Fellow): *An analytic characterization of surfaces of finite Lebesgue area* (II).

In abstract No. 64 of the January issue of this Bulletin, the author characterized non-degenerate surfaces of finite Lebesgue area. In the present paper,

the results are extended to arbitrary surfaces of finite area. First of all, it is shown that if S is represented on a hemicactoid (see the preceding abstract), $L(S)$ is the sum of the areas of the portions of S corresponding to the non-degenerate simple links of H (i.e., the Jordan regions of the plane set or the simple closed surfaces of the cactoids); the part of S corresponding to a portion of H containing no point of a non-degenerate simple link contributes nothing to $L(S)$ even though this portion may fill a cube. It is then shown that if S is of finite Lebesgue area, there exists a hemicactoid, H, each of whose simple closed surfaces is analytic except for two vertices, on which S can be represented continuously, the representation possessing the additional property that the part of S corresponding to any non-degenerate simple link of H is represented generalized conformally thereon, the area of that portion being given by the usual formula. (Received March 24, 1933.)

136. Dr. Gordon Pall: *Applications of the automorphs of sums of three squares to regularity of ternary quadratic forms.*

The forms of the rational automorphs of denominator p of $x^2+y^2+z^2$ are used to prove the regularity (hitherto in doubt) of certain positive ternary quadratic forms. (Received March 20, 1933.)

137. Dr. Gordon Pall: *On the relations between sums of three and four squares.*

There is established a precise correspondence between the representation of a prime p as a sum of four squares, the solutions of the congruence $x^2+y^2+z^2\equiv 0 \pmod{p}$, and the rational automorphs of denominator p of a sum of three squares. (Received March 20, 1933.)

138. Dr. Gordon Pall: *Towards a unification of the arithmetic of quadratic forms.*

It is generally agreed that the use of Kronecker binary quadratic forms leads to a theory of greater simplicity than that of the Gauss forms with their distinction between properly and improperly primitive forms. In this paper the various g.c.d.'s, invariants, and concomitants of any quadratic form in s variables ($s\geqq 2$) are so defined as to secure much greater simplicity in the general arithmetical theory of quadratic forms and to obviate the necessity of considering separately proper and improper classes or concomitant classes. (Received March 20, 1933.)

139. Professor A. A. Albert: *On universal sets of positive ternary quadratic forms.*

It is well known that no positive ternary quadratic form represents all positive integers. Hence a theory of universal positive ternaries cannot exist. In the present paper the author introduces the notion of universal sets of forms (such that every positive integer is represented by at least one form of the set) and chains of forms (universal sets with the property that the omission of a single form renders the set not universal). The chains of forms are to replace universal forms in the theory. In particular, chains of *sets of forms* are consid-

ered, each set consisting of *all* forms of a given determinant, and all such chains are completely determined. Every such chain is shown to consist of exactly two sets and moreover the author determines in this case precisely what integers are not represented by sets of *sets of forms* which are not universal. (Received March 22, 1933.)

140. Professor A. A. Albert: *A note on the Dickson theorem on universal ternaries.*

L. E. Dickson has proved (in his *Studies in the Theory of Numbers*, pp. 17–21) his theorem that every universal ternary quadratic form is a zero form. His proof is long, complicated, and very technical, and contains a considerable division into special cases. In the present note the author gives a short proof of Dickson's theorem. In fact a rational proof is given and it is shown that a non-singular ternary quadratic form with integer coefficients is a zero form *if and only if* it represents all integers for rational values of its variables. (Received March 22, 1933.)

141. Professor Dunham Jackson: *The convergence of series of orthogonal trigonometric sums.*

Some further theorems are obtained with regard to the convergence of the series discussed in a paper presented at the Los Angeles meeting of the Society, (Abstract No. 38–7–191). (Received March 24, 1933.)

142. Professor C. N. Moore: *On criteria for Fourier constants of L integrable functions.*

In a note published in the Proceedings of the National Academy of Sciences in May, 1932, the author obtained certain criteria which would assist in determining whether or not a set of constants $a_n (n=0, 1, 2, \cdots; a_n \to 0)$ were the Fourier constants of an L integrable function. In the present paper a more general criterion is obtained for the case where the constants do correspond to an integrable function. The added condition on the set of constants a_n is that the series $\Sigma n^r |\Delta^{r+1} a_n|$ should converge, where $0<r<1$ and $\Delta^{r+1}a_n$ represents the $(r+1)$st difference of the a's. This last criterion also includes a criterion due to Kolmogoroff (Bulletin de l'Académie Polonaise, 1922–23), which came to the attention of the author subsequent to the writing of the note referred to above. (Received March 22, 1933.)

143. Dr. Wilhelm Maier: *Lamé's functions and Lucas' numbers.*

Assume $r=0, 1, 2, \cdots$ a parameter; $S_r(u)=S_r$ an analytic function, $\partial S_r/\partial u = \sigma_r$. The system of three non linear differential equations $S_{r-1}S_r + S_{r+1}=0$; $S_r = S_{r+3}$ was treated by Euler, Jacobi and Poinsot. Let $n=2, 3, 4, \cdots$; and $\partial\alpha/\partial u = 0$; for appropriate parameters α_r, solutions of $S_{r-1}S_r + S_{r+1} + \alpha_r S_{r+1} = 0$; $S_r = S_{r+n}$ depend on Lamé's functions, involving their arithmetic properties. The classification of algebraic problems which arise leads to Lucas' numbers, namely to diophantine properties of $5^{1/2}$. (Received April 6, 1933.)

144. Professor Morris Marden: *Further mean-value theorems.*

The two principal theorems of the present note unite the results previously obtained by the author on *A generalization of Weierstrass' and Fekete's mean-value theorems* and *on the zeros of certain rational functions* (this Bulletin, vol. 38 (1932), pp. 434–441; Transactions of this Society, vol. 32 (1930), pp. 658–668). Among the new corollaries of the theorems are the following. *If, on the rectifiable curve C: $z=z(s)$, $g(z)$ is real, the point σ, $\sigma=(\int_a^b g(z)f(z)ds)/(\int_a^b|g(z)|ds)$, lies in the smallest convex region containing the two curves into which $w=\pm f(z)$ map C. If, in particular, $f(z)$ is a polynomial of degree n, it assumes this value σ at least once in the region consisting of all points from which C subtends an angle not less than $\pi/(2n)$.* The proofs of the two theorems follow methods of the above-mentioned articles. (Received March 23, 1933.)

145. Professor H. S. Wall: *On continued fractions which represent meromorphic functions.*

A sufficient condition (found by Van Vleck) that an infinite continued fraction (1) $[b_n z/1]_1^\infty$ shall represent a meromorphic function of z is that $\lim b_n=0$. The condition is necessary if the b_n are real and positive. On p. 309 of vol. 4 (1903) of the Transactions of this Society, Van Vleck stated that the condition is necessary when the b_n are real and $b_{2n}b_{2n+1}>0$. He proved that the roots of the denominator D_n of the nth convergent of (1) are real and simple, and that if $\cdots<x_{-3}<x_{-2}<x_{-1}<0<x_1<x_2<x_3<\cdots$ are the roots of D_{2n+1}, and those with increasing n approach distinct limiting positions not 0, then $\lim b_n=0$ and (1) is meromorphic. He thought, and so stated (without attempting proof) that the hypothesis of this theorem is fulfilled whenever (1) is meromorphic. The author supplements the above theorem by showing that (1) can be meromorphic when $\limsup|b_n|$ is >0, or ∞, and the b_n real, $b_{2n}b_{2n+1}>0$; and gives sufficient conditions applicable when the b_n are arbitrary real or complex numbers not 0. (Received March 16, 1933.)

146. Dr. P. W. Ketchum: *Expansions of two arbitrary analytic functions in series of rational functions.*

The particular series $\Sigma(\alpha_n+\beta_n x)F_n(x)$, where $F_n(x)=x^n/(1-x^{2n+1})$, converges to analytic functions inside a circle C of radius $R\leqq 1$ and outside a circle C' of radius $1/R$, and diverges between these circles and the unit circle. R is the smallest of the reciprocals of the superior limits of the nth roots of $|\alpha_n|$, $|\beta_n|$, and 1. Given any analytic function $f_1(x)$ one of whose singularities nearest to zero is at γ, and another function $f_2(x)$ one of whose farthest singularities is at γ', the α's and β's may be chosen so that the above series converges to $f_1(x)$ inside C and to $f_2(x)$ outside C', where the corresponding R is the smallest of $|\gamma|$, $1/|\gamma'|$ and 1. In general the coefficients α_n and β_n may be found in terms of certain contour integrals. Other analytic functions $F_n(x)$ satisfying the equation $F_n(1/x)=\pm xF_n(x)$ may be used, but the character of the convergence is in general not so simple. (Received March 22, 1933.)

147. Dr. E. S. Akeley: *An invariantive characterization of Riemannian geometries.*

Let $(\lambda_1 \cdots \lambda_n)$ be the characteristic values of λ for the pencil of forms $(G_{\mu\nu}-\lambda g_{\mu\nu})dx_\mu dx_\nu$ where $g_{\mu\nu}$ is a positive definite quadratic differential form in n dimensions, and $G_{\mu\nu}$ is its associated contracted Riemannian-Christoffel tensor. If $\partial(\lambda_1 \cdots \lambda_n)/\partial(x_1 \cdots x_n) \neq 0$, one can solve for the x's in terms of the λ's, and $(\lambda_1 \cdots \lambda_n)$ will form a canonical coordinate system. If $ds^2=\bar{g}_{\alpha\beta}d\lambda_\alpha d\lambda_\beta$, the $\bar{g}_{\alpha\beta}$ form a set of n^2 invariants which completely characterize the geometry. Many of the known Riemannian geometries are degenerate in the sense that the above Jacobian is zero. This is always true for $n=2$, and for spherically and axially symmetric solutions in higher dimensions. In order to find examples of non-degenerate geometries, an expansion of the type $g_{\mu\nu}=\delta_{\mu\nu}+\epsilon\gamma_{\mu\nu}+\cdots$, $G_{\mu\nu}=\epsilon H_{\mu\nu}+\cdots$ where ϵ is a parameter was used. The $H_{\mu\nu}$ were furthermore restricted to be linear functions of the $\lambda_i(i)$. The following results were obtained: (1) $n=2$, no solutions exist, (2) $n=3$, the solutions are uniquely determined and have been found, (3) $n>3$, a certain class of solution has been found. An expansion of the more general type $g_{\mu\nu}=C_{\mu\nu}+\epsilon\gamma_{\mu\nu}+\cdots$ has been used, where the $C_{\mu\nu}$ is a constant positive definite matrix, and corresponding theorems for this case are being obtained. (Received April 18, 1933.)

148. Professor T. R. Hollcroft: *Characteristics of multiple curves and their residuals.*

In this paper, relations between the respective characteristics of the two component curves of the complete intersection of two algebraic surfaces are derived, when one of the curves is i_1-fold on one surface and i_2-fold on the other. Formulas are also found for the characteristics of a complete intersection curve of given multiplicities on both surfaces. (Received March 24, 1933.)

149. Dr. Amos Black: *Types of involutorial space transformations associated with certain rational curves. Composite basis elements.*

Given a pencil of cubic surfaces $|F_3|$ which contains a rational curve r of order $m(m=2, 3, 4, 5)$. Make the surfaces of the pencil and the points of r projective. A point P in space will uniquely determine a surface F_3 and a point O on r. The line PO cuts F_3 in a third point P'. Thus on each line of the complex of lines which meet r is a pair of points P, P' in involution. The residual basis curve γ of $|F_3|$ is of order $9-m$. The case in which γ is irreducible was discussed in the Transactions of this Society, vol. 34 (1932), pp. 795–810. It is the purpose of the present paper to discuss the involutions when γ is composite, and when the projectivity between $|F_3|$ and r is specialized so as to reduce the degree of the involution. (Received March 21, 1933.)

150. Dr. J. M. Clarkson: *Involutorial line transformations defined by Cremona plane involutions.*

Consider in plane α a Cremona involution of one of the four fundamental types, and in plane β a Cremona involution of the same or of some other type.

An arbitrary line (y) of space, having Plücker coordinates $y_1, \cdots, y_6$, meets α in A whose image by the involution in α is A', and β in B whose image by the involution in β is B'. The line $(x) \equiv A'B'$ is the transform of (y), and the Plücker coordinates x_i are functions of y_j of degree equal to the sum of the orders of the two involutions. The various combinations of the four types of Cremona plane involutions are considered and the invariants and singular elements of the line transformations discussed. (Received March 18, 1933.)

151. Dr. S. S. Cairns: *On the triangulation of regular loci.*

In an n-space $(y_1, \cdots, y_n)$ we call an r-cell *regular* if it is definable by a homeomorphism $y_i = y_i(u_1, \cdots, u_r)$ with a convex polyhedral r-cell throughout some neighborhood of which the y's are defined, of class C', and possess some non-vanishing jacobian, $J(y_{i_1}, \cdots, y_{i_r})$. An r-manifold, M_r, means a compact connected point set consisting of (1) *interior points*, M_r', each with an r-cell for a neighborhood on M_r, and possibly (2) a *boundary*, B_{r-1}, any point, P, of which has for a neighborhood on M_r the closure of an r-cell with P on its boundary. *A completely regular r-manifold* is one satisfying the above definition read for regular r-cells, provided B_{r-1} is a finite set of distinct regular unbounded $(r-1)$-manifolds. (This assumes the definition of "regular k-manifold," see below, for $k<r$.) A *regular r-locus*, L_r, means a finite set of completely regular k-manifolds $(k=0, \cdots, r)$, called *elements* of L_r, any two of which (M_i, M_j) are so related that if M_j contains an interior point of M_i, then M_i is an element of the boundary of M_j. A *regular r-manifold* means a regular r-locus which is a manifold. The author triangulates L_r, and various modifications thereof, thus extending the theorem of Abstract No. 38–11–257 (to appear in the Annals of Mathematics). (Received March 22, 1933.)

152. Professor H. M. Gehman: *Concerning the definition of limit point.*

In this paper it is shown that the elementary theorems concerning limit point, closed set, and connected set are true, not only for the usual definition of limit point, but for any definition of limit point which has the following three properties: (1) A limit point of a set M is a limit point of any set containing M. (2) A limit point of the sum of two sets having no points in common is a limit point of at least one of the sets. (3) No point is a limit point of a set consisting of a single point. Certain groups of theorems require only one or two of these properties. In particular, a large number of theorems depend upon the first two properties only, and hence it is possible to consider a definition of limit point in a space consisting of a finite number of points. (Received February 14, 1933.)

153. Dr. I. J. Schoenberg: *A problem of Klein concerning the real roots of an algebraic equation.*

In a paper entitled *Geometrisches zur Abzählung der reellen Wurzeln algebraischer Gleichungen*, Gesammelte Werke, vol. 2, pp. 198–208, Felix Klein initiated the problem of comparing the efficiency of various theorems (Budan-Fourier, Descartes-Jacobi, Newton-Sylvester) which give upper bounds for the

number of real roots of an algebraic equation within a given interval. Klein has shown geometrically that for equations of the second degree the upper bound given by the theorem of Descartes-Jacobi never exceeds the bound furnished by the theorem of Budan-Fourier (see H. Weber, *Lehrbuch der Algebra*, 2d edition, vol. 1, pp. 354–357). H. Weber asks (loc. cit., p. 357) if this is true for equations of any degree. In the present paper this question is answered in the affirmative by analytical methods. It is also shown that the method of Descartes-Jacobi is always more efficient than a method due to Laguerre (Oeuvres, vol. 1, p. 10). Incidentally, this investigation shows that the theorems of Budan-Fourier and Laguerre are simple corollaries of Descartes' rule of signs. This paper will appear in the Mathematische Zeitschrift. (Received March 17, 1933.)

154. Dr. M. F. Deuring: *On the zeros of certain zeta functions.*

Riemann conjectured that all roots of the zeta function $\zeta(s)=\sum_{n=1}^{\infty}n^{-s}$, except the trivial ones at $-2, -4, \cdots$, have the real part $\frac{1}{2}$. This hypothesis, not yet proved, has been extended to more general functions, as they arise from the theory of algebraic numbers. Such functions are

$$Z_d(s) = \sum_{n,m\equiv-\infty}^{\infty}(n^2 + bnm + cm^2)^{-s},$$

where $x^2+bxy+cy^2$ is a definite quadratic form with discriminant $-d=b^2-4c<0$. These functions have many properties in common with $\zeta(s)$. For the functions $Z_d(s)$, the following theorem is proved in this paper: There exists a positive number c_0, such that all roots of $Z_d(s)$ in the region $0<t<d^{c_0}$ have the real part $\frac{1}{2}$. This is a sort of "approximation" of the Riemann hypothesis for these functions. (Received March 7, 1933.)

155. Mr. Casper Shanok: *Convex polyhedra and criteria for irreducibility.* Preliminary report.

This paper gives an application of Minkowski's theory of convex polyhedra to the construction of irreducibility criteria for polynomials in several variables. The theorems obtained generalize the results of Dumas on polynomials in one variable, and they contain as special cases various previously known theorems, e.g., a theorem by Glenn. The theory also gives a method of determining irreducibility in numerical cases through an analysis of the network obtained by the projection of the sides of the polyhedra on suitable planes. (Received March 23, 1933.)

156. Professor Edward Kasner: *Normal congruences of parabolas.*

The author discusses the conditions under which a congruence of ∞^2 parabolas, orthogonal to a given plane, constitute a normal congruence, that is, admit ∞^1 orthogonal surfaces. In addition, all *approximately* normal congruences (of various orders, in the sense defined by the author in his discussion of the converse of the theorem of Thomson and Tait, Transactions of this Society, vol. 11 (1910), p. 126, and the Princeton Colloquium Lectures) are determined explicitly, and appropriate geometric constructions are obtained. It turns out that when the approximation is of the third order, the congruence

is exactly normal. Each parabola cuts the given plane in two points and thus a definite transformation T is induced in the plane. A relation of T to Darboux transformations is obtained, and thereby connection is made with the theory of natural families of curves. Finally an analogue to Ribaucour's theorem (on normal congruences of circles) is derived in connection with the problem of finding all normal congruences in the system of ∞^5 vertical parabolas (see Princeton Colloquium Lectures, p. 64). (Received March 23, 1933.)

157. Mr. H. S. Grant: *Concerning powers of certain classes of ideals in a cyclotomic realm which give the principal class.*

It is a familiar theorem that if a prime $q \equiv 1$, mod n, then, in the cyclotomic number-realm determined by the nth roots of unity, the principal ideal $[q]$ is expressible as the product of $\phi(n)$ prime ideals of the first degree. Several results are known concerning the products of powers of these ideals which give the principal class. Results of this kind are obtained, for n a prime, by use of the Lagrange resolvent function and the associated Jacobi Ψ-function, and for n composite by use of their generalizations as given by Kummer and Stickelberger. The main purpose of this paper is to show, for n the power of an odd prime, that no matter how q^H be represented as the product of two conjugate imaginary ideal factors that are relatively prime, these factors are principal ideals, H being the "first-factor" of the class number of the realm. The method used is an extension of that employed by H. H. Mitchell (Transactions of this Society, 1918) for the case of an imaginary subfield of the cyclotomic field determined by the pth roots of unity, p an odd prime. It involves the use of the group-determinants of Frobenius (Berlin Berichte, 1896–1903). (Received March 23, 1933.)

158. Mr. W. S. Claytor: *Topological immersion of Peanian continua in a spherical surface.*

The author solves completely the problem of finding necessary and sufficient conditions under which a Peanian continuum will be topologically contained in a spherical surface. Kuratowski has handled the special case of this problem in which the Peanian continua contain at most a *finite* number of simple closed curves (Fundamenta Mathematicae, vol. 15, p. 271). In addition the present author obtains new, and very closely related, characterizations of both the simple closed surface and the closed 2-cell. (Received April 14, 1933.)

159. Dr. I. J. Schoenberg: *Convex domains and linear combinations of continuous functions.*

The object of this note is to give a simple geometric proof and a generalization of a theorem of L. L. Dines (Transactions of this Society, vol. 30 (1928), pp. 425–438) and some results of J. Favard (Bulletin de la Société Mathématique de France, vol. 59 (1931), pp. 229–255). One of the theorems obtained is as follows. Let $\phi_0(x) = 1$, $\phi_1(x)$, $\phi_2(x)$, $\cdots$ be a sequence of real continuous functions for $a \leqq x \leqq b (-\infty < a < b < +\infty)$, and $c_0 > 0$, c_1, c_2, $\cdots$ a given sequence of real constants. Every linear combination $\Phi(x) = a_0\phi_0(x) + a_1\phi_1(x) + \cdots + a_n\phi_n(x)$, whose coefficients satisfy the relation $a_0c_0 + a_1c_1 + \cdots$

$+a_nc_n=0$, will vanish at some point of $[a, b]$, if and only if the system $\int_a^b \phi_k(x)d\psi(x)=c_k(k=0, 1, 2, \cdots)$ admits a monotonic solution $\psi(x)$. This paper will appear in this Bulletin, in an early issue. (Received March 17, 1933.)

160. Dr. G. T. Whyburn: *Decompositions of continua by means of local separating points.*

If X is a subcontinuum of a continuum M, $L(X)$ will denote the set of local separating points of X. For each $p\varepsilon M$, there exist maximal subcontinua $B(p)$, $C(p)$, $D(p)$, and (for suitably restricted M) $E(p)$ containing p and such that $L[B(p)]$ and $L(M)\cdot C(p)$ are countable, $L[E(p)]$ and $L(M)\cdot D(p)$ are punctiform. No point belongs to two different sets B, C, D, or E. Thus we have decompositions of M into disjoint continua. The decompositions into sets $C(p)$ and $D(p)$ are upper semi-continuous; the hyperspaces C and D are regular curves; every subcontinuum of C contains uncountably many points of both $L(C)$ and $L(M)$; for no subcontinuum X of D is $X\cdot L(D)$ punctiform; thus no cyclic element of D has a continuum of condensation. $B(p)$ is the sum of all totally imperfect connected subsets containing p. If M is hereditarily locally connected, $E(p)$ exists and is the sum of all punctiform connected subsets containing p; in this case also the decompositions into sets $B(p)$ and $E(p)$ are upper semi-continuous; the hyperspaces B and E are regular curves, for each subcontinuum X of B, $L(X)$ is uncountable; B contains no totally imperfect connected set, for no subcontinuum X of E is $L(X)$ punctiform; hence E contains no punctiform connected set. (Received March 6, 1933.)

161. Professor Leonard Carlitz: *On a certain function connected with polynomials in a Galois field.*

The zeros of the function discussed in this paper are the set of polynomials in a single indeterminate over a fixed Galois field. Some of the properties of this function are developed, and applications are made to (1) the evaluation of sums of the type $\sum F^{-k}$ (summed over polynomials F); and (2) to two classes of congruences to a polynomial modulus. (Received March 23, 1933.)

162. Mr. R. H. Cameron: *New necessary and sufficient conditions that a transformation be almost periodic.*

In this paper a theorem is proved which enables one to tell whether a given transformation is almost periodic without first determining whether or not it possesses a single-valued inverse. (Received March 3, 1933.)

163. Dr. Hassler Whitney (National Research Fellow): *Differentiable functions defined on closed sets.*

Let A be a closed set in n-space E, let $f_{k_1\cdots k_n}(x_1, \cdots, x_n)$ be defined on A for $k_1+\cdots+k_n\leqq m$, and suppose that $f_{0\cdots 0}(x_1+h_1, \cdots, x_n+h_n)=\sum f_{k_1\cdots k_n}(x_1, \cdots, x_n)h_1^{k_1}\cdots h_n^{k_n}/(k_1!\cdots k_n!)+S(x_1, \cdots, x_n; h_1, \cdots, h_n)$ in A, where for every bounded subset B of A and $\epsilon>0$ there is a $\delta>0$ such that $|S(x_1, \cdots, x_n; h_1, \cdots, h_n)|<(\sum h_i^2)^{m/2}\epsilon$ for $(x_1, \cdots, x_n)$ in B if $(\sum h_i^2)^{1/2}<\delta$. Then if A satisfies certain conditions, for instance, if A is a closed domain with

a smooth boundary, we can extend the definition of the $f_{k_1 \cdots k_n}(x_1, \cdots, x_n)$ throughout E so that $(\partial/\partial x_i) f_{k_1 \cdots k_n} = f_{k_1 \cdots k_i+1 \cdots k_n}$ for $k_1 + \cdots + k_n < m$. This is a strengthening of a previous result of the author (see this Bulletin, January, 1933, p. 31). (Received March 22, 1933.)

164. Professor I. M. Sheffer: *Certain linear functional equations, and their relation to a regular singular point of a differential equation.*

Taking the operator (1) $J[y] = \sum_1^\infty a_n y^{(n)}(x)$ as fundamental element, we form the new operator (2) $L[y] \equiv \sum_0^\infty L_n(x) J^{(n)}[y]$, where $J^{(n)}[y] = J[J^{(n-1)}[y]]$, and $L_n(x)$ is a polynomial of degree $\leqq n$. The corresponding equation (3) $L[y] = \lambda y$ has a sequence of characteristic numbers λ_n for each of which there is a polynomial solution. We have elsewhere introduced the notion of *dual equation*. The dual of (3) is a linear differential equation (usually of infinite order). Let $H[y] = 0$ be a linear differential equation with either a regular singular point or an ordinary point at the origin. If we write (4) $H[y] = \lambda y$, an equation of type (3) is determined, of which (4) is the dual, and it is shown that the number of solutions of (4), analytic at the origin, is related to the number of polynomial solutions of (3), and the characteristic numbers of (3) and (4) are the same. (Received March 22, 1933.)

165. Dr. Jacob Sherman: *On the numerators of the convergents of the Stieltjes continued fractions.*

In this paper, which extends some of the results obtained by the author in a previous one (Transactions of this Society, vol. 35 (1933), pp. 65–84) are given (1) an explicit expression for the characteristic function $p_1(x)$ for the numerators of the convergents of the continued fraction "associated" with the classical orthogonal polynomials (Jacobi, Laguerre, Hermite); (2) a second-order non-homogeneous differential equation satisfied by the numerators of the odd convergents of the continued fraction "corresponding" to the polynomials of Jacobi and Laguerre; (3) the evaluation (obtained in the course of the computation) of certain singular definite integrals of type $J(x) = \int_a^b [(y-a)^{\alpha-1}(b-y)^{\beta-1}/(x-y)]dy$, for example, where $\alpha+\beta=1$, $\alpha, \beta > 0$, $(a, b) \equiv (-1, 1)$, $J(x) = \pi(-\cot(2n+1)\alpha\pi + i)(1+x)^{\alpha-1}(1-x)^{-\alpha}(-1<x<1)$, where n is chosen to obtain a particular αth root of -1. (Received March 21, 1933.)

166. Dr. Clement Winston: *On the mechanical quadratures formulas related to Hermite and Laguerre polynomials.*

The author considers the coefficients $H_{i,n}$ in the mechanical quadratures formula $\int_a^\infty f(x)p(x)dx = \sum_{i=1}^n H_{i,n} f(x_{i,n}) + R_n(f)$, $H_{i,n} = \int_a^\infty \{\phi_n(x)p(x)/[(x-x_{i,n})\phi_n'(x_{i,n})]\}dx$, $\phi_n(x_{i,n}) = 0$, where $\phi_n(x)(n = 0, 1, 2, \cdots)$ are the polynomials of Hermite and Laguerre with $p(x) = e^{-x^2}$, $e^{-x}x^{\alpha-1}(\alpha > 0)$, and $a = -\infty, 0$ respectively. In the case of Hermite polynomials it is shown, in a very simple way, that $K_n(x) = \sum_{i=0}^n \phi_i^2(x)$ is always increasing for $x > 0$, decreasing for $x < 0$, and hence is minimum at $x = 0$. This yields at once upper and lower bounds for the coefficients $H_{i,n}$ (see an article by the author to appear soon in the Annals of Mathematics, where similar results have been ob-

tained in a different way). Moreover, the same considerations lead to the following interesting property of the coefficients: $\{H_{i,n}\}$ separate $\{H_{i,n+1}\}$. Similar results, although not so complete, are obtained for Laguerre polynomials. (Received March 20, 1933.)

167. Dr. Hillel Poritsky: *Application of analytic function theory to two-dimensional elasticity problems.*

In this paper analytic function theory is applied to the solution of two-dimensional elasticity problems that reduce to finding functions satisfying the equation (1) $\nabla^4 F=0$, $\nabla^4=(\partial^2/\partial x^2+\partial^2/\partial y^2)^2$, and proper boundary conditions. The general solution of (1) in the form $F=R[f(z)+\bar{z}g(z)]$, where $z=x+iy$, $\bar{z}=x-iy$, f and g are analytic, and R denotes "the real part of," is employed. The boundary conditions are expressed in terms of f and g, and solutions are given for several boundaries. In particular, Green's functions and "reflections" of applied point forces in plane and circular boundaries are discussed. (Received March 24, 1933.)

168. Dr. Selby Robinson (National Research Fellow): *The χ^2 test of goodness of fit.*

In 1922 R. A. Fisher proposed a modification in the use of the χ^2 test in case the hypothesis being tested was secured by a method which involved the computation of c statistics from the data. A satisfactory proof of Fisher's theory was given by J. Neyman and E. S. Pearson in the special case in which the sample is assumed to be indefinitely large, the c statistics are efficient, and a third condition is also satisfied. The present paper deals with a coin tossing experiment in which there are seventy samples of 128 items each. Seventy χ_1^2's were computed and found to be distributed in accordance with Fisher's theory. (Received March 23, 1933.)

169. Professor C. R. Adams: *On non-factorable transformations of double sequences.*

The matrix $\|a_{mnkl}\|$ ($m, n, k, l=1, 2, 3, \cdots$, $a_{mnkl}=0$ for $k>m$ or $l>n$ or both) defines a transformation of double sequences (A): $y_{mn}=\sum_{k,l=1}^{m,n} a_{mnkl}x_{kl}$, which is called factorable when and only when $a_{mnkl}=a'_{mk}\cdot a''_{nl}$ ($m, n, k, l=1, 2, 3, \cdots$). Lösch (Mathematische Zeitschrift, vol. 34 (1931), pp. 281–290) has proved a theorem which, after a slight extension by the present writer, may be given the following form: A factorable transformation (A), regular for bounded sequences, is regular for all sequences having bounded A-transforms. Agnew (American Journal, vol. 54 (1932), pp. 648–656) has shown that such a transformation is regular for all sequences whose A-transforms are ultimately bounded (that is, $\lim\sup_{m,n\to\infty}|y_{mn}|<\infty$), thus enlarging the class to its maximum. The purpose of the present note is to show that *neither theorem remains valid when the hypothesis of factorability is deleted.* This is done by exhibiting the example $a_{mn1l}=\delta_{mn}\delta_{nl}/n$, $\delta_{ii}=1$, $\delta_{ij}=0$ for $i\neq j$; $a_{mnkl}=1/(mn)$ ($k=2, 3, 4, \cdots$; $m, n, l=1, 2, 3, \cdots$): the transformation (A) thus defined carries the null sequence $x_{1l}=l$, $x_{kl}=0$ ($k=2, 3, 4, \cdots$; $l=1, 2, 3, \cdots$) into the sequence $y_{mn}=\delta_{mn}$, bounded but divergent. This example shows also that a non-

factorable transformation (A) may satisfy conditions far more stringent than those which are necessary and sufficient for regularity for bounded sequences and still not be regular for all sequences having bounded A-transforms. (Received April 14, 1933.)

170. Professor J. F. Ritt: *Integral functions obtained by compounding polynomials.*

Let $P_1(z), \cdots, P_n(z)$ be a sequence of polynomials whose degrees do not exceed a fixed integer m and which, ordered in ascending powers of z, start with the term z. This paper considers the sequence of polynomials $Q_n(z)$ defined by $Q_1(z)=P_1(z)$; $Q_{n+1}(z)=Q_n[P_{n+1}(z)]$, $n=1, 2, \cdots$, and also the sequence of $R_n(z)$ defined by $R_1(z)=P_1(z)$; $R_{n+1}(z)=P_{n+1}[R_n(z)]$, $n=1, 2, \cdots$. Conditions are given under which these sequences converge to integral functions. (Received April 14, 1933.)

171. Dr. B. F. Kimball: *Application of Bernoulli polynomials of negative order to differencing.* Second paper.

The author extends the results obtained in his previous paper to the case of unequal difference intervals, real and complex. The following theorem is proved. *Let $w_1, w_2, \cdots, w_n$ denote the n real positive difference intervals of $B_{2m}^{-n}(x, w_1w_2\cdots w_n)$. Set $t_n=(w_1+w_2+\cdots+w_n)/2$. Let $\sum^r w_k$ represent the sum of r difference intervals w_k taken from the above group, none being repeated. If r is an integer (or zero) such that $t_n-\sum^r w_k\geq 0$ for all possible sums $\sum^r w_k$ taken from the above group; then $B_{2m}^{-n}(-t_n, w_1w_2\cdots w_n)\leq\{n(n-1)\cdots(n-r)/[(n+2m)(n-1+2m)\cdots(n-r+2m)]\}t^{2m}$, the equality sign not applying when $n>2$.* This theorem enables us to test the rapidity of convergence of a series expansion of a difference in terms of the Bernoulli polynomials. If all the difference intervals are pure imaginaries so that $\omega_k=iw_k$, $B_{2m}^{-n}(-\tau_n, \omega_1\omega_2\cdots\omega_n)=(-1)^m B_{2m}^{-n}(-t_n, w_1w_2\cdots w_n)$, $\tau_n=(\omega_1+\omega_2+\cdots+\omega_n)/2$. In the case of complex difference intervals, the following relation simplifies matters: $|B_{2m}^{-n}(-\tau_n, \omega_1\omega_2\cdots w_n)|\leq B_{2m}^{-n}(-t_n, w_1w_2\cdots w_n)$, where $w_k=|\omega_k|$, $\tau_n=(\omega_1+\omega_2+\cdots+\omega_n)/2$. (Received March 3, 1933.)

172. Professor J. A. Shohat: *On some applications of the Tchebycheff inequality for definite integrals.*

Consider Taylor's formula for a given function $f(x)$, and write its remainder in the well known form of a definite integral. To the latter, where under the integral sign we find a product of two functions, we apply the Tchebycheff inequality for definite integrals, assuming that *a certain derivative $f^{(\nu)}(x)$ is monotonic in the interval under consideration.* Thus we obtain various estimates of the said remainder leading to some interesting inequalities for the classical cases (e^x, $\sin x$, etc.). (Received March 7, 1933.)

173. Dr. S. S. Wilks (International Research Fellow): *A criterion for testing the hypothesis of mutual independence of a set of normal multivariate populations.*

A criterion of the Neyman-Pearson λ type is found by the method of maximum likelihood for testing the hypothesis H that an n-variate sample Σ of N

individuals comes from a population which is composed of a set of p mutually independent sub-populations $\{\pi_t\}$ when it is assumed that Σ has been drawn from some n-variate normal population. The criterion is expressible in the form $\lambda_H = D/\Pi_{i=1}^{p} d_i$, where D is the determinant of variances and covariances of all n variates of Σ and d_t is the principal minor of D of order $n_t (\Sigma_t^p{}_{=1} n_t = n)$ whose elements are the sample values of the variances and covariances belonging to π_t. An expression is found for the sampling distribution of λ_H when H is true, which yields relatively simple probability integrals for certain values of p and the n_t. The limiting form of the distribution of λ_H for large samples is shown to be a Pearson Type III frequency function. (Received March 21, 1933.)

174. Professor J. L. Walsh: *Line integrals as a measure of polynomial approximation to analytic functions.*

Let C be a point set of the z-plane not separating the plane but consisting of a finite number of rectifiable Jordan arcs and Jordan regions bounded by rectifiable curves, let $w=\phi(z)$ map (conformally but not necessarily uniformly) the complement of C onto $|w|>1$ so that the points at infinity correspond to each other, and let C_ρ denote the curve or curves $|\phi(z)|=\rho>1$. If $f(z)$ is analytic on C, there is a largest finite or infinite $R>1$ such that $f(z)$ (or its analytic extension) is single-valued and analytic interior to C_R. Then any sequence of polynomials of best approximation to $f(z)$ in the sense of least pth powers ($p>0$) measured by a line integral over the boundary of C with a positive continuous (this restriction may be lightened) norm function converges to $f(z)$ interior to C_R, uniformly on any closed point set interior to C_R, and converges uniformly in no region containing in its interior a point of C_R. (Received March 27, 1933.)

175. Dr. Stanislaw Saks: *Addition to the note "On some functionals."*

In the present note the author extends to completely additive functions of sets in abstract spaces some results of the paper mentioned in the title. (Received April 7, 1933.)

176. Mr. L. B. Robinson: *On equations in mixed differences.* Part IV.

Consider the determinant $|\Sigma_{i-j}+a_{ij}|$. The sum of the a's is a converging double series. When $i-j$ is negative or positive and greater than m, Σ vanishes. When $i-j$ is zero, $\Sigma=1$. In all other cases the sigmas represent the symmetric functions of the roots of a polynomial, all of whose roots are absolutely less than unity. Under the above assumptions, by addition and subtraction of rows or columns we can reduce the above determinant to a form whose convergence has been demonstrated by Poincaré. Consider now an equation previously studied by the author (see this Bulletin, vol. 32, p. 316). When all the a's are inferior in absolute value to unity, then with the help of the above determinant we show that a general solution of this equation exists, that is, it depends on a variable parameter. If one or more of the a's are greater in ab-

solute value than unity, the existence of a singular solution can be proved. Our equation is a normal form to which other equations are reducible. (Received April 1, 1933.)

177. Professor R. L. Jeffery: *Derivatives with respect to functions of bounded variation.*

In a previous paper by the author (Transactions of this Society, vol. 34, p. 645) the derived numbers of $F(x)$ with respect to a function of bounded variation $\alpha(x)$ were defined for functions F with discontinuities of the first kind only. The present paper obtains the distribution of the values of these derived numbers for any finite function F, whether it be measurable relative to α or not. Theorems on approximate derivatives with respect to α are obtained, also on λ-derivatives with respect to α analogous to those obtained by Burhill, Haslam-Jones, and Besicovitch (Proceedings of the London Mathematical Society, (2), vol. 32, No. 5) for ordinary derivatives. Further results are the following: a method for determining F when a derived number of F with respect to α is given and finite at each point; the necessary and sufficient conditions that F be an indefinite integral with respect to α, or a total indefinite integral with respect to α. (Received March 14, 1933.)

178. Professor C. R. Adams and Mr. J. A. Clarkson: *On definitions of bounded variation for functions of two variables.*

Several definitions have been given of conditions under which a function of two real variables shall be said to be of bounded variation. Of these definitions six are usually associated with the names of Vitali, Hardy, Arzelà, Pierpont, Fréchet, and Tonelli, respectively. A seventh, formulated by Hahn and attributed by him to Pierpont, we readily prove equivalent to Pierpont's definition. For convenience, let the classes of functions satisfying the respective definitions be denoted by V, H, A, P, F, and T; in addition, let the class of continuous functions be designated by C, and let a product (such as $A \cdot T$) stand for the subclass of functions common to the two or more classes named. The fundamental relations of inclusiveness between these classes are as follows: (1) $P > A > H$, $F > V > H$, $T > H$. We easily show that any two classes not named in the same part of (1) overlap, and we investigate the extent of the common part of two or more classes (such as $A \cdot T$). The same questions are studied when only continuous functions are admitted to consideration, in which case we have, in contrast with (1), $T \cdot C > P \cdot C > A \cdot C > H \cdot C$, $F \cdot C > V \cdot C > H \cdot C$. (Received February 16, 1933.)

179. Professor J. A. Shohat: *On the transfinite diameter for a linear interval.*

It is known (Stieltjes) that the zeros of Jacobi polynomials $\phi_n(x; \alpha, \beta)$ possess certain extremal properties (resulting from the differential equation which they satisfy). We let α and β approach zero, and thus get an explicit expression for the polynomial of degree n, whose zeros maximize the absolute value of the Vandermonde determinant $D_n(x_1, \cdots, x_n)$, the points x_i belonging to a finite interval (which, for $n \to \infty$, leads to the "transfinite diameter"

(Fekete) for the said interval). This expression is simpler and exhibits more plainly the properties of the said x_i than that given by I. Schur (Mathematische Zeitschrift, vol. 1 (1918)). We further show how these x_i can be applied to interpolation and mechanical quadratures, improving a result due to Fekete. (Received March 7, 1933.)

180. Professors Einar Hille and J. D. Tamarkin: *Questions of relative inclusion in the domain of Hausdorff matrices.*

In the present paper we investigate the question of when $[H, q_1(u)] \supset [H, q_2(u)]$, i.e., when a Hausdorff definition of summability includes another. In particular, we have found necessary and sufficient conditions for $[H, q(u)] \supset (C, \alpha)$ and for $[H, q(u)] \supset (E_p)$. It turns out that if these relations hold simultaneously, then $[H, q(u)] \supset (C, \alpha)$ (E_p) as well. We have found some sufficient conditions ensuring that an analytic function of a Hausdorff matrix be another such matrix. As a particular case we obtain a generalization of a theorem of I. Schur (Mathematische Annalen, vol. 74 (1913), pp. 453–456) for all Hausdorff matrices. In addition we have found a number of new proofs for already known inclusion theorems; thus, for instance, still another proof for the equivalence of the definitions of Cesàro and Hölder is obtained. (Received March 25, 1933.)

181. Professors Einar Hille and J. D. Tamarkin: *On the summability of Fourier series.* IV.

In the present paper we extend our investigations on the summability of Fourier series to the general class of "radial" matrices introduced by R. Schmidt. To each matrix A of this type there corresponds a "mass-function" $q_A(u)$ of bounded variation over $(0, \infty)$, such that $q_A(0)=0$. For such matrices the question of their (F)-effectiveness is answered completely by the following theorem. In order that the matrix A should sum to $f(x)$ the Fourier series of an arbitrary integrable function $f(x)$, at each point x such that $f(x+t)+f(x-t)-2f(x)\to 0$ as $t\to 0$, it is necessary and sufficient that the cosine Fourier transform of $q(u)$ be integrable over $(0, \infty)$. The continuity of $q(u)$ for $u>0$ results as a necessary condition of the (F)-effectiveness. The class of matrices in question embraces practically all regular definitions of summability known in the literature and the method used allows us to decide with great ease the question of the (F)-effectiveness of various important transformations, such, for instance, as those of Hausdorff, Le Roy, de la Vallée Poussin, and others, which in many cases caused considerable difficulties. (Received March 25, 1933.)

182. Professor J. L. Walsh: *On the location of the critical points of Green's function for a plane region.*

Any finite number of mutually exterior Jordan curves C_k can be simultaneously uniformly approximated by a lemniscate. Green's function G_1, with pole at infinity for the region C exterior to these curves, is thus approximated by Green's function G_2 with pole at infinity for the exterior of the lemniscate; G_2 is of the form $a \log |p(z)| + b$, where $p(z)$ is a polynomial. The critical points

of G_2 are the roots of the derivative of $p(z)$, and the roots of $p(z)$ can be chosen interior to the C_k. Well known theorems (Lucas, Jensen, Walsh) on the roots of the derivative of a polynomial thus lead to results on the critical points of G_1. For example, if C is an infinite region whose boundary is limited, then all critical points of the (assumed existent) Green's function for C with pole at infinity lie in the smallest convex region containing the boundary of C. (Received April 15, 1933.)

183. Mr. R. L. Peek, Jr.: *Some new theorems on limits of variation.*

The theorems presented in this paper permit the evaluation of upper limits to the relative frequency of occurrence of members of a frequency distribution differing from the arithmetic mean by more than a specified amount. They are therefore similar to the Tchebycheff and Camp-Meidell inequalities, but differ from them in involving not only the standard deviation, but the mean deviation as well, and consequently afford, in general, much closer limits. A preliminary lemma shows that the standard deviation of the distribution of absolute deviations from the mean (σ_1), is equal to the square root of the difference between the squares of the standard and mean deviations ($\sigma_1^2=\sigma^2-\bar{x}_1^2$). If ρ is the ratio of the mean and standard deviations ($\bar{x}_1/\sigma$), it is shown that the relative frequencies of values differing from the mean by more than $t\sigma$ is less than $t\rho^2/t^2-2t\rho+1$. For a continuous frequency distribution, subject to the same limitations as apply to the Camp-Meidell inequality, the relative frequency of values differing from the mean by more than $t\sigma$ is shown to be less than $4(1-\rho^2)/9(t-\rho)^2$. A further theorem gives another expression for such upper limits in terms of the mean and standard deviations and certain additional parameters. (Received April 28, 1933.)

184. Professor W. D. Baten: *The probability law for the sum of n independent variables when each is subject to the law* $(1/2h)$ *sech* $(\pi x/2h)$.

The probability law for the sum of the n independent variables is found by Dodd's method, which leads to certain definite integrals involving the hyperbolic secant raised to the nth power. These definite integrals are evaluated for $n=2, 3, 4, 5, 6$ by integrating over certain contours in the complex plane and allowing two parallel boundaries to approach infinity. Mathematical induction is employed to evaluate the integrals for any values of n, the proof of which depends on proving that certain terms in a Laurent expansion are integers and on the evaluation of a certain definite integral, which is implied by ideas relating to ${}_nC_r$. An approximation for the law is obtained when n approaches infinity. Wallis' theorem is obtained as a by-product together with explicit expressions for the derivatives $d^{2n}(hw/\sin hw)^{2n}/dw^{2n}$ and $d^{2n}(w/\sin w)^{2n}/dw^{2n}$. (Received April 27, 1933.)

185. Professor M. S. Knebelman: *A canonical form for a set of vectors.*

In this paper it is shown how a coordinate system may be obtained in which the components of a given set of vectors take a particularly simple form. If $S(\leqq n)$ independent contravariant vectors in a V_n are given, such that the completed set contains $r(\geqq s, \leqq n)$ independent ones, then a coordinate system may be found in which all components after the rth are zero and the Kth components of the mth vector for $K>m$ vanishes on a given subspace V_{n-m+1}. If a complete set of r independent covariant vectors in V_n is given, then a coordinate system is obtained in which all components after the rth are zero, the remaining ones being functions of at most r variables such that the Kth component of the mth vector for $K<m$ vanishes on a given subspace V_{n-m+1}. (Received April 27, 1933.)

186. Professor M. S. Knebelman: *On Finsler spaces.*

In a recent paper (Comptes Rendus, Feb., 1933) E. Cartan showed how the geometry of Finsler spaces may be brought more closely into line with Riemannian geometry by changing the law of parallelism. He points out the advantages of his definition of parallel displacement over that of L. Berwald. This note shows that the advantage is not entirely on the side of Cartan and that some of the tensor invariants obtained by him do not have the geometrical significance usually associated with them. (Received April 27, 1933.)

187. Mr. W. O. Gordon: *A neighborhood treatment of general topological spaces.*

In this paper it is shown that the most general topological space (in the sense of Fréchet) can be treated as a neighborhood space by using a definition of limit point in terms of neighborhood which is slightly different from the usual ones. Necessary and sufficient conditions on neighborhoods in order that the space have certain well known properties are then derived. (Received May 1, 1933.)

188. Dr. D. H. Lehmer: *On imaginary quadratic fields whose class number is unity.*

This paper gives the results of an investigation of the magnitude of the class number $h(\sqrt{-D})$ of the quadratic field $K(\sqrt{-D})$ with special reference to the cases in which $h=1$. Gauss noted that $h=1$ for $D=1, 2, 3, 7, 11, 19, 43, 67$, and 163, and conjectured that only a finite number of such D's exist, after an examination of all D's less than 3000. By showing that $x^2+x+(D+1)/4$ is composite for some $x<15$, Dickson was able to prove that $h>1$ for $163<D<1{,}500{,}000$. The present investigation enables one to make this assertion for $163<D<5{,}000{,}000{,}000$. The method differs from that of Dickson, Rainich, and Nagel, and depends on the calculation of the least number N_p of the form $8n+3$ for which $-N_p$ is a quadratic non-residue of all odd primes $\leqq p$. The method not only shows that $h>1$, but also gives a lower limit for h. The examination of this large range of values of D was made with the author's number sieve, which sifted the determinants at a rate of 100,000 a second. (Received May 6, 1933.)

189. Professor M. A. Basoco: *On the element of decomposition of a doubly periodic function of the second kind.*

The element of decomposition for a doubly periodic function of the second kind, in the sense of Hermite, may be written in terms of the Weierstrass sigma function in the form $\sigma(u+v)e^{\rho u}/\sigma(u)\sigma(v)$. It may be shown that it satisfies a differential equation of the Lamé type. For suitable values of ρ special functions of importance in the applications are obtained. In particular, the expansion in powers of u of this function for $\rho=x-\zeta(v)$, where $\zeta(v)$ is the Wierstrass zeta function, is discussed. The coefficients in the development are Appell polynomials in x, which satisfy a certain differential equation and can be constructed by recurrence. Certain other Appell polynomials associated with these are also obtained and discussed. (Received May 5, 1933.)

190. Mr. G. D. Nichols: *On the arithmetized Fourier developments of certain doubly periodic functions of the second kind.*

This paper derives the explicit arithmetized Fourier series expansions for the functions $\Phi_{\alpha\beta\gamma}(x, y)=\theta_1^{12}[\theta_\alpha^2(x+y)/\theta_\beta^2(x)\cdot\theta_\gamma^2(y)]$, where the θ's are the elliptic theta functions of Jacobi, x and y are independent complex variables, and (α, β, γ) are certain sixteen triads out of the possible sixty-four which can be selected from the numbers 0, 1, 2, 3. The method used follows closely that of M. A. Basoco (this Bulletin, vol. 38 (1932), p. 560) and is a generalization of that due to Teixeira (Journal fur Mathematik, vol. 125 (1901), p. 301). The developments have applications in the theory of quadratic forms. The expressions $\theta_i^{12}(y)/\theta_k^2(y)$ and $\theta_i^{11}(y)/\theta_k^2(y)$ $(i=0, 1, 2, 3)$ occur in the above developments. The explicit arithmetized expansions are also worked out for these, and it is believed they are an addition to the lists of such expansions given by Bell (Messenger of Mathematics, vol. 53 (1924), p. 166) and others. (Received May 4, 1933.)

191. Mr. Nathan Jacobson: *Non-commutative polynomials and cyclic algebras.*

Let $F'(x)$ denote the domain of polynomials in x with coefficients in a cyclic field $F'=F(a)$ of order $r>1$, for which $xa=\bar{a}x$ where $a=\theta(a)$ gives a generating automorphism of the Galois group of F' over F. Consider the cyclic algebra $F(\Pi)$ as the algebra of residues of the polynomials of $F'[x]$ modulo $\Pi=x^r-\gamma(\gamma\neq 0)$. With any polynomial, P, associate a residue algebra $F(P)$ restricted to the sub-domain $F(P)$ of $F'(x)$ of polynomials A for which $PA=A'P$. If $\Pi=PP^*$, P irreducible, then $F(P)$ is isomorphic to the division algebra part of $F(\Pi)$ in the Wedderburn decomposition. If $\Pi=P_1P_2\cdots P_s$ is a decomposition of Π into irreducible factors, each P has the same degree t and the order of the division algebra part of $F(\Pi)$ is t^2. These results reduce the problem of determining the structure of $F(\Pi)$ to a factorization problem in $F'(x)$. They give immediately the theorem of Hasse, that a necessary and sufficient condition that $F(\Pi)$ be matric is that γ be the norm of an element of F'. From Wedderburn's norm conditions is obtained the generalization that if q is the smallest integer such that γ^q is the norm of an element of F', then the division algebra part of $F(\Pi)$ has order $\geqq q^2$. The theorems of Brauer concerning the exponent of a cyclic algebra then follow. (Received May 3, 1933.)

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